

wishmom version 3.0

Matlab programs for the moments of β -Wishart and inverse β -Wishart matrices

User manual

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1 Introduction

This is the manual for version 3.0 of the Matlab programs accompanying

Hillier, G. and Kan, R. M. (2024). On the expectations of equivariant matrix-valued functions of Wishart and inverse Wishart matrices. *Scandinavian Journal of Statistics*, 51, 1–27 (henceforth HK),

and the companion note

Hillier, G. and Kan, R. M. (2026). Recurrence relations for the moments of β -Wishart and inverse β -Wishart matrices (henceforth HK26),

which extends the results of HK from the real and complex Wishart distributions to the quaternion Wishart distribution and, for the scalar moments $\mathbb{E}[p_\kappa(W)]$ and $\mathbb{E}[p_\kappa(W^{-1})]$, to the general β -Wishart distribution ($\beta > 0$).

The programs compute, for $W \sim \mathcal{W}_m^\beta(n, \Sigma)$,

$$\mathbb{E}\left[\prod_{j=1}^r \text{tr}(W^j)^{f_j} W^{iw}\right] \quad \text{and} \quad \mathbb{E}\left[\prod_{j=1}^r \text{tr}(W^{-j})^{f_j} W^{-iw}\right], \quad (1)$$

either numerically for given n and Σ (`wishmom.m`, `iwishmom.m`) or as explicit polynomials/rational functions of n (`wishmom_sym.m`, `iwishmom_sym.m`), together with all the coefficient matrices (D_k , $\mathcal{C}_k$, $\mathcal{H}_k$, D_{-k} , $\tilde{D}_k$, $\tilde{\mathcal{C}}_k$, $\tilde{\mathcal{H}}_k$) of HK. The algorithms are the recurrences of HK, Theorem 1 (real case) and Theorem 4 (complex case), in the one-parameter form of HK26, Theorem 1.1.

What is new in version 3.0

- **Bug fix.** The existence test in `iwishmom.m` (versions 1.1–2.3) was too strict: it returned NaN whenever $\tilde{n} \leq 2k$, whereas the moment of total degree k exists if and only if $\tilde{n} > (k-1)\alpha$ (Section 2.4). The same error is present in version 1.1 of the R package `wishmom`.
- **Bug fix.** `wishmom_rec.m` was correct only for $\alpha = 1, 2$; it now handles every α .
- **New functions** `wishmom_sym.m` and `iwishmom_sym.m` (analytical expressions without the Symbolic Math Toolbox), `iwish_exist.m`, `wm_index.m`, `wm_polystr.m`, and the test script `wishmom_test.m`.
- **Robustness.** `dkimapr.m`, `qkn_coeffr.m` and `iwish_psr.m` verify their output at a test value of $\tilde{n}$ and warn if precision has been lost; `qk_coeff.m`, `qkn_coeff.m` and their `_sym` versions accept a missing `alpha`; the degenerate case $k = 0$ is handled.
- Documentation corrected throughout; this manual.

Requirements

Matlab R2020b or later (`pagetimes` is used by `qk_coeff.m`, `qkn_coeff.m`, `wish_ps.m`, `iwish_ps.m`). The Symbolic Math Toolbox is required only by the routines whose names end in `_sym` that return symbolic matrices (`qk_coeff_sym`, `qkn_coeff_sym`, `wish_ps_sym`, `iwish_ps_sym`), by `dkimap.m` when `n1` is omitted, and by the scripts `example1.m`–`example6.m`. `wishmom_sym.m` and `iwishmom_sym.m` do *not* need it. After unpacking, run `wishmom_test` to check the installation.

2 Mathematical background

2.1 The β -Wishart distribution and the parameter α

Let $\beta \in \{1, 2, 4\}$ and let $\mathbb{F}$ be $\mathbb{R}$, $\mathbb{C}$ or the quaternions $\mathbb{H}$ accordingly. $W \sim \mathcal{W}_m^\beta(n, \Sigma)$ denotes an $m \times m$ Hermitian W over $\mathbb{F}$ with density

$$f(W) \propto |W|^{\frac{(n-m+1)\beta}{2}-1} \text{etr}\left(-\frac{\beta}{2}\Sigma^{-1}W\right), \quad n > m-1,$$

so that $\mathbb{E}[W] = n\Sigma$; for integer n , $W = \Sigma^{1/2} X X^* \Sigma^{1/2}$ with the β real coordinates of each entry of the $m \times n$ matrix X i.i.d. $N(0, 1/\beta)$. Following the R package, the programs are parametrised by

$$\alpha = \frac{2}{\beta} : \quad \alpha = 2 \text{ (real, default),} \quad \alpha = 1 \text{ (complex),} \quad \alpha = \frac{1}{2} \text{ (quaternion),}$$

and the inverse moments are expressed in terms of

$$\tilde{n} = n - m + 1 - \alpha \quad (\text{n1 in the programs}), \quad (2)$$

which is $n - m - 1$ in the real case and $n - m$ in the complex case. Other values $\alpha > 0$ are accepted; see Section 2.5.

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ we write $p_\lambda(W) = \prod_i \text{tr}(W^{\lambda_i})$. In the programs a partition is specified in *frequency notation* by the vector $f = (f_1, \dots, f_r)$, where f_j is the number of parts equal to j ; thus $\mathbf{f} = [2 \ 0 \ 1]$ is $\lambda = (3, 1, 1)$ and $\prod_j \text{tr}(W^j)^{f_j} = \text{tr}(W)^2 \text{tr}(W^3)$. The quantities (1) are $\mathbb{E}[W^{iw} p_\lambda(W)]$ and $\mathbb{E}[W^{-iw} p_\lambda(W^{-1})]$, of total degree $k = |\lambda| + iw = \sum_j j f_j + iw$.

2.2 The recurrences

HK26, Theorem 1.1: for $r \geq 0$ and every partition λ ,

$$\begin{aligned} \mathbb{E}[W^{r+1} p_\lambda(W)] &= [n + (\alpha - 1)r] \Sigma \mathbb{E}[W^r p_\lambda(W)] + \sum_{j=1}^r \Sigma \mathbb{E}[W^{r-j} p_j(W) p_\lambda(W)] \\ &\quad + \alpha \sum_{i=1}^{\ell(\lambda)} \lambda_i \Sigma \mathbb{E}[W^{r+\lambda_i} p_{\lambda_{(i)}}(W)], \end{aligned} \quad (3)$$

$$\begin{aligned} \Sigma^{-1} \mathbb{E}[W^{-r} p_\lambda(W^{-1})] &= [\tilde{n} - (\alpha - 1)r] \mathbb{E}[W^{-(r+1)} p_\lambda(W^{-1})] - \sum_{j=1}^r \mathbb{E}[W^{-r-1+j} p_j(W^{-1}) p_\lambda(W^{-1})] \\ &\quad - \alpha \sum_{i=1}^{\ell(\lambda)} \lambda_i \mathbb{E}[W^{-r-1-\lambda_i} p_{\lambda_{(i)}}(W^{-1})], \end{aligned} \quad (4)$$

with $\mathbb{E}[W] = n\Sigma$, $\mathbb{E}[W^{-1}] = \Sigma^{-1}/\tilde{n}$, where $\lambda_{(i)}$ is λ with its i th part removed. For $\alpha = 2$ these are (28)–(29) of HK; for $\alpha = 1$ they are (86)–(87).

2.3 Coefficient matrices

Let $\mathcal{L}_k(W)$ be the stacked matrix of HK, eq. (5): the blocks $p^{(k-r)}(W) \otimes W^r$ for $r = k, k-1, \dots, 1$, where $p^{(j)}(W)$ lists $p_\kappa(W)$ over the partitions $\kappa \vdash j$ in *reverse lexicographic* order (the order produced by `ip_desc.m`: for $j = 4$, $(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$). $\mathcal{L}_k$ has $d_k = \sum_{i=0}^{k-1} \pi(i)$ blocks; the position of the block $W^{iw} p_\lambda(W)$ is returned by `wm_index(f, iw)`, and the position of p_κ in $p^{(k)}$ by `wm_index(f, 0)`. With $\mathcal{Q}_k = \mathbb{E}[\mathcal{L}_k(W)]$, $q_k = \mathbb{E}[p^{(k)}(W)]$, and the analogous $\tilde{\mathcal{Q}}_k$, $\tilde{q}_k$ for W^{-1} ,

$$\mathcal{Q}_k = (\mathcal{C}_k \otimes I_m) \mathcal{L}_k(\Sigma), \quad q_k = \mathcal{H}_k p^{(k)}(\Sigma), \quad \tilde{\mathcal{Q}}_k = (\tilde{\mathcal{C}}_k \otimes I_m) \mathcal{L}_k(\Sigma^{-1}), \quad \tilde{q}_k = \tilde{\mathcal{H}}_k p^{(k)}(\Sigma^{-1}),$$

and the recurrences give (HK, Prop. 3 and eq. (65); HK26, Section 5)

$$\mathcal{Q}_{k+1} = (D_k \otimes \Sigma) \begin{bmatrix} \mathcal{Q}_k \\ q_k \otimes I_m \end{bmatrix}, \quad \begin{bmatrix} (I \otimes \Sigma^{-1}) \tilde{\mathcal{Q}}_k \\ \tilde{q}_k \otimes \Sigma^{-1} \end{bmatrix} = (D_{-k} \otimes I_m) \tilde{\mathcal{Q}}_{k+1}, \quad (5)$$

$$D_{-k} = \tilde{n}I - (D_k - nI), \quad \tilde{D}_k = D_{-k}^{-1}, \quad (6)$$

$$\mathcal{C}_{k+1} = D_k \text{diag}(\mathcal{C}_k, \mathcal{H}_k) = \text{diag}(\mathcal{C}_k, \mathcal{H}_k) D_k, \quad \mathcal{H}_k = (\alpha k)^{-1} D_{k,21} \mathcal{C}_k D_{k,12}, \quad \mathcal{C}_1 = \mathcal{H}_1 = n, \quad (7)$$

$$\tilde{\mathcal{C}}_{k+1}^{-1} = D_{-k} \text{diag}(\tilde{\mathcal{C}}_k^{-1}, \tilde{\mathcal{H}}_k^{-1}), \quad \tilde{\mathcal{H}}_k^{-1} = (\alpha k)^{-1} D_{k,21} \tilde{\mathcal{C}}_k^{-1} D_{k,12}, \quad \tilde{\mathcal{C}}_1^{-1} = \tilde{\mathcal{H}}_1^{-1} = \tilde{n}. \quad (8)$$

Here D_k is $d_{k+1} \times d_{k+1}$ with $n + (\alpha - 1)r$ on the diagonal of the block $W^{r+1} p_\lambda \leftarrow W^r p_\lambda$, and constants elsewhere; `dkmap(k, alpha)` returns $D_k - nI$. The entries of $\mathcal{C}_k$, $\mathcal{H}_k$ are polynomials in n of degree $\leq k$ without constant term, the entries of $\tilde{\mathcal{C}}_k^{-1}$, $\tilde{\mathcal{H}}_k^{-1}$ are polynomials in $\tilde{n}$ of the same kind, and the entries of $\tilde{\mathcal{C}}_k$, $\tilde{\mathcal{H}}_k$, $\tilde{D}_k$ are rational functions of $\tilde{n}$ with a common denominator (Section 2.4). All of them are universal: they depend on α (and n or $\tilde{n}$) but not on m or Σ . Every row of D_k sums to $n + k\alpha$, of $\mathcal{C}_k$ to $\alpha^k(n/\alpha)_k$, of D_{-k} to $\tilde{n} - k\alpha$, and of $\tilde{\mathcal{C}}_k$ to $1/\prod_{j=0}^{k-1}(\tilde{n} - j\alpha)$.

2.4 Existence of the inverse moments and denominators

The moment $\mathbb{E}[W^{-iw} p_\lambda(W^{-1})]$ of total degree k exists if and only if (HK26, Prop. 6.1)

$$\frac{n - m + 1}{\alpha} > k \quad \Longleftrightarrow \quad \tilde{n} > (k - 1)\alpha \quad \Longleftrightarrow \quad n > m - 1 + k\alpha. \quad (9)$$

In the real case this is $n - m + 1 > 2k$ (HK, Remark 1); in the complex case $n - m + 1 > k$; in the quaternion case $n - m + 1 > k/2$. `iwishmom.m` returns NaN when (9) fails, and the test itself is available as `iwish_exist(n,m,k,alpha)`.

The common denominator of the entries of $\tilde{\mathcal{H}}_k$ and $\tilde{\mathcal{C}}_k$ is the polynomial in $\tilde{n}$ returned by `denpoly(k, alpha)`: its roots are the numbers $i - 1 - j\alpha$ over the cells $(i, j + 1)$ of the partitions of k (each root with the largest multiplicity it has in a single partition), so the largest root is $(k - 1)\alpha$, the threshold in (9). The common denominator of $\tilde{D}_k$ is $\prod(\tilde{n} - \alpha j + i)$ over $\{(i, j) : i, j \geq 0, (i + 1)(j + 1) \leq k + 1\} \setminus \{(0, 0)\}$, the α -contents of the cells that can be added to a partition of k (HK26, Theorem 5.11); its largest root is $k\alpha$. The cell $(0, 0)$ is addable only to the empty diagram and must be excluded here; `unid.m`, which builds the root list, excludes it. Note the difference between the two lists: `denpoly` ranges over *all* cells of the partitions of k (so it contains the content 0 once, coming from $\tilde{\mathcal{C}}_1 = 1/\tilde{n}$), while `unid` ranges over the *addable* cells.

2.5 General $\beta > 0$

For $\alpha \notin \{2, 1, \frac{1}{2}\}$ there is no Wishart matrix over a division algebra, but the joint distribution of the eigenvalues of W is defined for every $\beta > 0$ (the β -Laguerre ensemble with general scale matrix), and the scalar moments $\mathbb{E}[p_\kappa(W)]$, $\mathbb{E}[p_\kappa(W^{-1})]$ are well-defined symmetric functions of the eigenvalues of Σ . The programs run the recurrences with the given α and return $\mathcal{H}_k^{(\alpha)}$, $\tilde{\mathcal{H}}_k^{(\alpha)}$; HK26, Theorems 7.3 and 7.12 prove that these are correct for every $\alpha > 0$ and every k (the proof uses only the joint eigenvalue density, through the Dunkl operators of type A). For such α the matrix-valued outputs ($iw > 0$) are the Dunkl-operator moments of HK26, Remark 7.4; they coincide with $\mathbb{E}[W^{iw} p_\lambda(W)]$ only when $\alpha \in \{2, 1, \frac{1}{2}\}$. Note also that the routines are designed for the “nice” values $\alpha \in \{2, 1, \frac{1}{2}\}$; for other α the polynomial coefficients are not integers and the rational-polynomial routines may lose precision for large k (a warning is issued).

3 Function reference

All functions accept `alpha` as an optional argument with default 2. In what follows k always denotes the total degree.

3.1 Main functions

`Q = wishmom(n,S,f,iw,alpha)` returns $\mathbb{E}[\prod_j \text{tr}(W^j)^{f_j} W^{iw}]$ for $W \sim \mathcal{W}_m^\beta(n, S)$: a scalar if $iw = 0$, an $m \times m$ matrix otherwise. `f` is the frequency vector (a scalar `f` means $f_1 = f$); `iw` defaults to 0. `S` may be complex Hermitian when `alpha = 1`. The result is a polynomial in `n`, so `n` need not be an integer. Uses `wish_psn` (`iw = 0`) or `qk_coeffn`.

```
>> n = 20; S = [25 49; 49 109];
>> wishmom(n,S,4) % E[tr(W)^4], real Wishart
ans = 8.7050836561920e+13
>> wishmom(n,S,[2 0 1],2,2) % E[tr(W)^2 tr(W^3) W^2], real Wishart
ans =
    9.0394621085225e+23    1.9569478210099e+24
    1.9569478210099e+24    4.2587139040121e+24
>> Sc = [25 49+2i; 49-2i 109];
>> wishmom(n,Sc,[2 0 1],0,1) % E[tr(W)^2 tr(W^3)], complex Wishart
ans = 2.0781257772076e+17
```

`Q = iwishmom(n,S,f,iw,alpha)` returns $\mathbb{E}[\prod_j \text{tr}(W^{-j})^{f_j} W^{-iw}]$, or NaN (a NaN matrix) if (9) fails. Uses `iwish_psn` or `qkn_coeffn`.

```
>> iwishmom(n,S,2) % E[tr(W^{-1})^2]
ans = 6.6808921404258e-04
>> iwishmom(n,S,[2 0 1],2,2) % E[tr(W^{-1})^2 tr(W^{-3}) W^{-2}]
ans =
    1.3284335517473e-10   -6.1016922930468e-11
   -6.1016922930468e-11    2.8242915865361e-11
>> iwishmom(n,Sc,[2 0 1],0,1) % complex Wishart
ans = 1.1798504023078e-08
>> iwishmom(4,S,1) % m=2, n=4: n1=1>0, E[tr(W^{-1})] exists
ans = 1.2727 ...
>> iwishmom(3,S,1) % n1=0: does not exist
ans = NaN
```

`out = wishmom_sym(f,iw,alpha)` returns the analytical expression of $\mathbb{E}[\prod_j \text{tr}(W^j)^{f_j} W^{iw}]$ as a structure. For $iw = 0$ the expression is $\sum_{\kappa \vdash k} h_\kappa p_\kappa(\Sigma)$ and the structure has fields `kappa` (cell array of partitions), `coef` (cell array of coefficient vectors of h_κ in descending powers of n , constant term included), `poly` (strings). For $iw > 0$ the expression is $\sum_{i=1}^k [\sum_{\rho \vdash k-i} c_{i,\rho} p_\rho(\Sigma)] \Sigma^i$ and the fields are `i`, `rho`, `coef`, `poly`. In both cases `latex` and `text` contain the whole expression as a string.

```
>> o = wishmom_sym(4); o.latex
(48n)p_{(4)}(\Sigma)+(32n^2)p_{(3,1)}(\Sigma)+(12n^2)p_{(2,2)}(\Sigma)
+(12n^3)p_{(2,1,1)}(\Sigma)+(n^4)p_{(1,1,1,1)}(\Sigma)
>> o = wishmom_sym([1 1],1); o.text % E[tr(W)tr(W^2)W]
(24n^2+24n)S^4+(4n^3+4n^2+16n)p_{(1)}(S)S^3+[(2n^3+2n^2+8n)p_{(2)}(S)+(6n^2)p_{(1,1)}(S)]S^2
```

```

+[(4n^3+4n^2)p_(3)(S)+(n^4+n^3+4n^2)p_(2,1)(S)+(n^3)p_(1,1,1)(S)]S
>> o = wishmom_sym(4,0,1); o.latex % complex Wishart
(6n)p_{(4)}(\Sigma)+(8n^2)p_{(3,1)}(\Sigma)+(3n^2)p_{(2,2)}(\Sigma)+(6n^3)p_{(2,1,1)}
(\Sigma)+(n^4)p_{(1,1,1,1)}(\Sigma)

```

(The h_k printed by version 1.1 of the R package are these divided by n , because its formatting routine treats the coefficient of n as a constant term; the numerical routines of the R package are unaffected.)

`out = iwishmom_sym(f,iw,alpha)` returns the analytical expression of $\mathbb{E}[\prod_j \text{tr}(W^{-j})^{f_j} W^{-iw}]$ as a ratio of polynomials in $\tilde{n}$: the fields `coef`, `poly` are the numerators, `den/denpoly` the common denominator, `exists` the condition (9), and `latex`, `text` the whole expression ($\tilde{n}$ is written `n1` in `text`).

```

>> o = iwishmom_sym(2); o.text % E[tr(W^{-1})^2]
[(2)p_(2)(S^{-1})+(n1-1)p_(1,1)(S^{-1})]/(n1^3-n1^2-2n1)
>> o = iwishmom_sym(4,0,1); o.latex % E[tr(W^{-1})^4], complex Wishart
[(30\tilde{n})p_{(4)}(\Sigma^{-1})+(16\tilde{n})^2-24)p_{(3,1)}(\Sigma^{-1})+(3\tilde{n})^2+18)p_{(2,2)}(\Sigma^{-1})
+(6\tilde{n})^3-24\tilde{n})p_{(2,1,1)}(\Sigma^{-1})+(\tilde{n})^4-8\tilde{n})^2+6)p_{(1,1,1,1)}(\Sigma^{-1})]
/(\tilde{n})^8-14\tilde{n})^6+49\tilde{n})^4-36\tilde{n})^2)
>> o.exists
n1 > 3 (n1 = n-m+1-alpha, alpha = 1)

```

`Q = wishmom_rec(n,S,f,iw,alpha)` computes the same quantity as `wishmom` directly by the recurrence (3). It is provided for illustration only and is slow for large k .

`[ok,thr] = iwish_exist(n,m,k,alpha)` returns `ok = true` if the inverse moments of degree k exist, and the threshold `thr = m-1+k*alpha` ($n > \text{thr}$ is required).

3.2 Coefficient matrices: moments of W

`D = dkmap(k,alpha)` returns $D_k - nI_{d_{k+1}}$ (HK, Prop. 3 and Appendix B, with the entries $2\kappa_i$ replaced by $\alpha\kappa_i$ and r by $(\alpha - 1)r$).

```

>> dkmap(2) % real: D_2 = ans + n*I, cf. HK eq. (36)
      2      1      1      0
      2      1      0      1
      4      0      0      0
      0      4      0      0
>> dkmap(2,1/2) % quaternion
     -1.0    1.0    1    0
      0.5   -0.5    0    1
      1.0    0.0    0    0
      0.0    1.0    0    0

```

`c = qk_coeff(k,alpha,n)` returns $\mathcal{C}_k$. Without `n`, `c` is a $d_k \times d_k \times k$ array whose j th slice holds the coefficients of n^{k+1-j} ($j = 1: n^k$; $j = k: n^1$; there is no constant term). With `n` (numeric or symbolic) the polynomials are evaluated. `qk_coefn(k,alpha,n)` does the evaluation directly in

double precision (faster; `n` required); `qk_coeff_sym` works in symbolic arithmetic (exact for every k , but slow).

```
>> c = qk_coeff(2); c(:,:,1), c(:,:,2)      % C_2 = [n^2+n, n; 2n, n^2]
      1      0      1      1
      0      1      2      0
>> qk_coeffn(2,2,7)
      56      7
      14     49
```

`h = wish_ps(k,alpha,n)`, `wish_psn`, `wish_ps_sym` return $\mathcal{H}_k(\pi(k) \times \pi(k))$, with the same conventions. The row of $\mathcal{H}_k$ for κ is at position `wm_index(f,0)`.

3.3 Coefficient matrices: moments of W^{-1}

`Di = dkimap(k,alpha,n1)` returns $\tilde{D}_k = D_{-k}^{-1}$; without `n1` the result is symbolic in `n1`.

`[D,a] = dkimapr(k,alpha)` returns $\tilde{D}_k$ as a ratio of polynomials in $\tilde{n}$ without the symbolic toolbox: `D(:,:,j)` is the coefficient matrix of $\tilde{n}^{L-j}$ in the numerator ($L = \text{size}(D,3)$, the last slice is the constant term) and `a` holds the coefficients of the denominator in descending powers (constant term included).

```
>> [D,a] = dkimapr(1); D(:,:,1), D(:,:,2), a
      1      0      0      1      1     -1     -2
      0      1      2     -1
```

i.e. $\tilde{D}_1 = \begin{bmatrix} \tilde{n} & 1 \\ 2 & \tilde{n}-1 \end{bmatrix} / (\tilde{n}^2 - \tilde{n} - 2)$, HK eq. (70).

`c = qkn_coeff(k,alpha,n1)`, `qkn_coeffn`, `qkn_coeff_sym` return $\tilde{C}_k^{-1}$ as polynomials in `n1` (slices $\tilde{n}^k, \dots, \tilde{n}^1$).

`[C,a] = qkn_coeffr(k,alpha)` returns $\tilde{C}_k$ itself, as a ratio of polynomials in $\tilde{n}$: `C(:,:,j)` are the numerator coefficient matrices in descending powers (last slice = constant term), and `a` the coefficients of the common denominator in descending powers, *omitting* the zero constant term (the denominator is $\sum_j a_j \tilde{n}^{L+1-j}$ with $L = \text{length}(a)$).

`iwish_ps`, `iwish_psn`, `iwish_ps_sym` `h = iwish_ps(k,alpha,n1)` and its variants return $\tilde{\mathcal{H}}_k^{-1}$; `[H,a] = iwish_psr(k,alpha)` returns $\tilde{\mathcal{H}}_k$ as numerator/denominator polynomials with the same conventions as `qkn_coeffr`.

```
>> [H,a] = iwish_psr(2); H(:,:,1), H(:,:,2), a
      1      0      0      1      1     -1     -2
      0      1      2     -1
```

i.e. $\tilde{\mathcal{H}}_2 = \begin{bmatrix} \tilde{n} & 1 \\ 2 & \tilde{n}-1 \end{bmatrix} / (\tilde{n}^3 - \tilde{n}^2 - 2\tilde{n})$.

`y = denpoly(k,alpha)` returns the coefficients (descending powers, constant term omitted) of the common denominator of $\tilde{\mathcal{H}}_k$ and $\tilde{C}_k$; e.g. `denpoly(3)` is `[1 -3 -8 12 16]`, i.e. $\tilde{n}^5 - 3\tilde{n}^4 - 8\tilde{n}^3 + 12\tilde{n}^2 + 16\tilde{n}$.

`y = unid(k,alpha)` returns the distinct values y such that $\tilde{n} + y$ are the eigenvalues of D_{-k} (`unid(3,2) = [-6 -4 -2 -1 1 2 3]`); used by `dkimapr`.

3.4 Auxiliary functions

`S = ip_desc(k)` lists the partitions of k in reverse lexicographic order, one per row, padded with zeros.

`[ind,k] = wm_index(f,iw)` gives the position of $W^{iw} \prod_j \text{tr}(W^j)^{f_j}$ in $\mathcal{L}_k$ (if $iw > 0$) or of p_λ in $p^{(k)}$ (if $iw = 0$), and the degree k . E.g. `wm_index([2 0 1],2) = 16` (the block $W^2 p_{(3,1,1)}$ in $\mathcal{L}_7$).

`s = wm_polystyr(c,v)` formats a polynomial with coefficients c (descending powers) in the variable v .

`wishmom_test` runs the consistency checks listed in its header (explicit formulae against recurrence, rational-polynomial routines against double-precision routines, symbolic against numeric, the commutation $D_k \text{diag}(\mathcal{C}_k, \mathcal{H}_k) = \text{diag}(\mathcal{C}_k, \mathcal{H}_k) D_k$, row sums, Monte Carlo checks of the unbiased estimators (63)–(64) and (84)–(85) of HK, and the existence test).

3.5 Example scripts

`example1.m–example6.m` reproduce the tables of $D_k, \mathcal{H}_k, \mathcal{C}_k, \tilde{D}_k, \tilde{\mathcal{H}}_k, \tilde{\mathcal{C}}_k$ for $k \leq 6$ (symbolically) and time the routines for $k \leq 20$; their output is in the corresponding `.out` files.

4 Conventions and pitfalls

- **Ordering.** Partitions are always in reverse lexicographic order; the blocks of $\mathcal{L}_k$ are ordered by decreasing power of W . Use `wm_index` rather than counting by hand.
- **Polynomial storage.** The 3-d arrays returned by `qk_coeff`, `wish_ps`, `qkn_coeff`, `iwish_ps` store the coefficients of $n^k, \dots, n^1$ (or $\tilde{n}^k, \dots, \tilde{n}^1$) and have *no* constant-term slice; the arrays returned by `dkimapr`, `qkn_coeffr`, `iwish_psr` *do* include the constant term. The denominators returned by `qkn_coeffr`, `iwish_psr`, `denpoly` omit the (zero) constant term; the one returned by `dkimapr` includes it.
- **Precision.** The coefficients are integers (for $\alpha \in \{2, 1\}$) or dyadic rationals ($\alpha = \frac{1}{2}$) that grow quickly with k ; double precision is exact up to $k \approx 11$ for the denominator polynomials and somewhat further for the numerators. For larger k use the `_sym` routines, or use the numeric routines (`wishmom`, `iwishmom`, `*n.m`), which are accurate for much larger k because they evaluate at the given n rather than manipulating polynomials.
- **Complex Σ .** For $\alpha = 1$, Σ may be Hermitian complex. Quaternion scale matrices cannot be entered directly; a real symmetric Σ is a valid quaternion Hermitian matrix, and for a general quaternion Σ one can use the power sums $p_j(\Sigma) = \frac{1}{2} \text{tr} \varphi(\Sigma)^j$ of its $2m \times 2m$ complex representation $\varphi(\Sigma)$ together with the coefficients from `wishmom_sym/iwishmom_sym`.
- **Timing.** On a laptop, `qk_coeffn(15,2,n)` and `qkn_coeffn(15,2,n1)` take about 0.2s, `qkn_coeffr(10,2)` and `dkimapr(10,2)` a few hundredths of a second; the $k \leq 20$ timings of HK, Table 1 are reproduced by `example2.m–example6.m`.

5 References

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