An explicit expression for the expectation of a product of quadratic forms in multivariate normal random variables

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Abstract

We present an explicit formula for the expectation of a product of k quadratic forms in a multivariate normal random vector with an arbitrary mean vector and a positive semidefinite covariance matrix. The proposed expression contains no repeated terms, so it has the fewest possible number of terms, and the coefficient attached to every term is given in closed form. We also derive an equivalent expression, indexed by set partitions rather than permutations, that is far more efficient to enumerate. The results are extended to Hermitian quadratic forms in complex normal random vectors, and they are shown to deliver compact expressions for the product moments of central and noncentral, real and complex Wishart distributions. Finally, we obtain an explicit expression that accommodates repeated quadratic forms in the product, in which case the number of distinct terms is further reduced. Complete proofs of all results are provided, and a set of accompanying computer programs implements every formula in the paper.

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1. Introduction

Let $\boldsymbol{z} = (z_1, \ldots, z_m)^{\mathrm{T}} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ be an m-dimensional normal random vector with mean $\boldsymbol{\mu}$ and positive semidefinite covariance matrix $\boldsymbol{\Sigma}$, and let $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ be symmetric $m \times m$ matrices. The object of interest in this paper is an explicit expression for

$$\mathrm{E}\left[\prod_{i=1}^{k} \boldsymbol{z}^{\mathrm{T}} \boldsymbol{A}_i \boldsymbol{z}\right]. \tag{1}$$

Expectations of this type arise throughout multivariate statistics and econometrics, for example in the study of moments of sample variances and covariances, of ratios of quadratic forms, of test statistics, and of Wishart matrices.

In the central case $\boldsymbol{\mu} = \boldsymbol{0}_m$, an explicit expression for (1) was developed in a series of papers by Kumar [13], Magnus [16, 17], and Don [3]. For the noncentral case $\boldsymbol{\mu} \neq \boldsymbol{0}_m$, however, no explicit expression is available in the literature except for small k. What is available for general k are recursive methods. Bao and Ullah [1] and Hillier, Kan, and Wang [9] present recurrence relations for (1), and Kan [10] expresses (1) as a linear combination of k-th moments of single quadratic forms, each of which can be computed recursively. These methods permit efficient numerical evaluation of (1), and, after expansion, they can also deliver analytical expressions. The expressions so obtained, however, contain many repeated terms, and the coefficient attached to each term must itself be determined recursively. For small k the output can be simplified by hand, but an explicit formula for general k has remained unknown in the noncentral case. This paper fills that gap.

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Our main result (Proposition 1) is an explicit expression for (1) in which every term is distinct, so the expression is in its simplest possible form, and every coefficient is an explicit power of two. The result is obtained by combining a classical expression for the moments of a single quadratic form with a polarization argument, which we spell out in detail. The expression of Don [3] for the central case is recovered as an immediate corollary. Because the terms of the main expression are indexed by permutations satisfying certain admissibility conditions, enumerating them requires screening all $k!$ permutations of $\{1, \ldots, k\}$. Our second result (Proposition 2) removes this bottleneck: it re-indexes the same expression by set partitions of $\{1, \ldots, k\}$, whose number, the Bell number B_k, is far smaller than $k!$.

We then develop three extensions. First, we obtain the analogous expressions for products of Hermitian quadratic forms in complex normal random vectors (Proposition 3). A subtle point emerges here: the real part must be taken termwise, and overlooking this point has led to incorrect expressions in the literature (see Section 6). Second, we show that the product moments $\mathrm{E}[\prod_{i=1}^k \mathrm{tr}(\boldsymbol{W}\boldsymbol{A}_i)]$ of central and noncentral, real and complex Wishart matrices $\boldsymbol{W}$ follow directly from our formulas (Corollaries 2 and 3), yielding expressions that are more compact than those in the literature and correcting published errors. Third, we treat the case in which the quadratic forms in the product are raised to powers, $\mathrm{E}[\prod_{i=1}^k (\boldsymbol{z}^\mathrm{T}\boldsymbol{A}_i\boldsymbol{z})^{s_i}]$ (Proposition 4). When some matrices are repeated, the number of distinct terms drops substantially, and our expression exploits this reduction. Along the way we make use of a closed-form formula for the number of bracelets with fixed content (Lemma 8), a count obtained recently by Hadjicostas [8], for which we include a short self-contained proof.

Following the requirements of the journal, complete proofs of every lemma, proposition, and corollary are collected in Appendix A. A set of MATLAB programs implementing all of the formulas in this paper is available at `https://www-2.rotman.utoronto.ca/~kan/research.htm`.

The remainder of the paper proceeds as follows. Section 2 introduces notation and preliminary counting results. Section 3 presents an explicit expression for $\mathrm{E}[(\boldsymbol{z}^\mathrm{T}\boldsymbol{A}\boldsymbol{z})^k]$. Section 4 contains the main result for $\mathrm{E}[\prod_{i=1}^k \boldsymbol{z}^\mathrm{T}\boldsymbol{A}_i\boldsymbol{z}]$ and its central-case corollary. Section 5 gives the computationally efficient version. Sections 6 and 7 treat complex normal vectors and Wishart product moments. Section 8 handles repeated quadratic forms, and Section 9 concludes.

2. Notation and preliminaries

Throughout, $\boldsymbol{z} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}$ positive semidefinite, and $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ are symmetric $m \times m$ matrices; T denotes transposition. We refer to a finite sequence of positive integers $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_p)$ as a *word*, and we write $\ell(\boldsymbol{\nu}) = p$ for its length. Words serve as indices of the two families of scalars from which all of our expressions are built:

$$\tau_{\boldsymbol{\nu}} = \mathrm{tr}\left(\boldsymbol{A}_{\nu_1}\boldsymbol{\Sigma}\boldsymbol{A}_{\nu_2}\boldsymbol{\Sigma}\cdots\boldsymbol{A}_{\nu_p}\boldsymbol{\Sigma}\right), \tag{2}$$

$$\theta_{\boldsymbol{\nu}} = \boldsymbol{\mu}^\mathrm{T}\boldsymbol{A}_{\nu_1}\boldsymbol{\Sigma}\boldsymbol{A}_{\nu_2}\boldsymbol{\Sigma}\cdots\boldsymbol{\Sigma}\boldsymbol{A}_{\nu_p}\boldsymbol{\mu}. \tag{3}$$

When $\boldsymbol{\nu} = (i)$ is a singleton we simply write $\tau_i = \mathrm{tr}(\boldsymbol{A}_i\boldsymbol{\Sigma})$ and $\theta_i = \boldsymbol{\mu}^\mathrm{T}\boldsymbol{A}_i\boldsymbol{\mu}$.

Two invariance properties of (2) and (3) drive all of the combinatorics in this paper. Write $\boldsymbol{\nu}^R = (\nu_p, \ldots, \nu_1)$ for the reversal of $\boldsymbol{\nu}$ and $\sigma\boldsymbol{\nu} = (\nu_2, \ldots, \nu_p, \nu_1)$ for its cyclic rotation. Since the trace is invariant under cyclic permutations, and since transposing a product of symmetric matrices reverses its order,

$$\tau_{\boldsymbol{\nu}} = \tau_{\sigma\boldsymbol{\nu}} = \tau_{\boldsymbol{\nu}^R}, \qquad \theta_{\boldsymbol{\nu}} = \theta_{\boldsymbol{\nu}^R}. \tag{4}$$

Thus $\tau_{\boldsymbol{\nu}}$ depends on $\boldsymbol{\nu}$ only through the equivalence class of $\boldsymbol{\nu}$ under rotations and reversal, and $\theta_{\boldsymbol{\nu}}$ only through the pair $\{\boldsymbol{\nu}, \boldsymbol{\nu}^R\}$.

In combinatorics, an equivalence class of words under rotations is called a *necklace*, and an equivalence class under rotations and reversal is called a *bracelet*. We identify each class with its lexicographically smallest member and use the following canonical representatives:

(i) a word $\boldsymbol{\nu}$ is a *canonical θ-word* if it does not exceed its reversal in lexicographic order, i.e., $\boldsymbol{\nu} \preceq \boldsymbol{\nu}^R$;

(ii) a word $\boldsymbol{\nu}$ is a *canonical τ-word* (a bracelet) if it is the lexicographically smallest member of its equivalence class under rotations and reversal.

When the entries of $\boldsymbol{v}$ are distinct, being a canonical τ-word is equivalent to the requirements that (a) v_1 is the smallest entry, and (b) $v_2 < v_p$ whenever $p \geq 3$; being a canonical θ-word is equivalent to $v_1 < v_p$ whenever $p \geq 2$. For example, $(1,2,4)$ is a canonical τ-word, whereas $(1,4,2)$, $(2,1,4)$, $(2,4,1)$, $(4,1,2)$, and $(4,2,1)$ all represent the same bracelet and are not canonical.

Our first lemma counts the distinct values of θ and τ that can be formed from a word with distinct entries.

Lemma 1. *Let $\boldsymbol{v}$ be a word with i distinct entries. The number of distinct θ-terms that can be formed by arranging the entries of $\boldsymbol{v}$ is*

$$n_\theta(i) = \begin{cases} 1, & i = 1, \\ i!/2, & i \geq 2, \end{cases} \tag{5}$$

and the number of distinct τ-terms is

$$n_\tau(i) = \begin{cases} 1, & i \leq 2, \\ (i-1)!/2, & i \geq 3. \end{cases} \tag{6}$$

For a positive integer k, we write $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_r) \vdash k$ for a partition of k with parts $\kappa_1 \geq \cdots \geq \kappa_r > 0$, and $r = \ell(\boldsymbol{\kappa})$ for its number of parts. The *frequency representation* of $\boldsymbol{\kappa}$ is $\boldsymbol{\kappa} = 1^{f_1} 2^{f_2} \cdots k^{f_k}$, where f_i is the number of parts of $\boldsymbol{\kappa}$ equal to i; we collect the frequencies in $\boldsymbol{f} = (f_1, \dots, f_k)$, so that $\sum_{i=1}^k i f_i = k$ and $\sum_{i=1}^k f_i = r$. As partitions and frequency vectors are in one-to-one correspondence, we use $\boldsymbol{\kappa}$ and $\boldsymbol{f}$ interchangeably.

Finally, let $\mathcal{B}_k$ denote the collection of set partitions of $\{1, \dots, k\}$, i.e., decompositions of $\{1, \dots, k\}$ into disjoint nonempty blocks. The cardinality of $\mathcal{B}_k$ is the Bell number B_k, and an efficient algorithm for enumerating $\mathcal{B}_k$ is given by Nijenhuis and Wilf [20]. For $\boldsymbol{\pi} \in \mathcal{B}_k$ we write $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \dots, \boldsymbol{\pi}_r)$ for its blocks, $\ell(\boldsymbol{\pi}) = r$ for the number of blocks, and, as no significance attaches to the order of the blocks, we list them in ascending order of block size.

3. The moments of a single quadratic form

We first record an explicit expression for $\mathrm{E}[(\boldsymbol{z}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{z})^k]$, the special case of (1) with $\boldsymbol{A}_1 = \cdots = \boldsymbol{A}_k = \boldsymbol{A}$. For a symmetric matrix $\boldsymbol{A}$, define

$$\tilde{\tau}_i = \mathrm{tr}\left((\boldsymbol{A\Sigma})^i\right), \qquad \tilde{\theta}_i = \boldsymbol{\mu}^{\mathrm{T}} (\boldsymbol{A\Sigma})^{i-1} \boldsymbol{A\mu}, \qquad i \geq 1. \tag{7}$$

Lemma 2. *Let $\boldsymbol{z} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}$ positive semidefinite and let $\boldsymbol{A}$ be a symmetric matrix. The j-th cumulant of $\boldsymbol{z}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{z}$ is*

$$\psi_j = 2^{j-1}(j-1)!\left(\tilde{\tau}_j + j\tilde{\theta}_j\right), \qquad j \geq 1. \tag{8}$$

Lemma 2 is classical when $\boldsymbol{\Sigma}$ is positive definite (see, e.g., Mathai and Provost [19, p. 55]); the proof in Appendix A covers the positive semidefinite case as well. Combining Lemma 2 with the classical expression of moments in terms of cumulants yields the following explicit expression, which is essentially Lemma 3 of Magnus [18].

Lemma 3. *Let $\boldsymbol{z} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}$ positive semidefinite and let $\boldsymbol{A}$ be a symmetric matrix. Then*

$$\mathrm{E}\left[(\boldsymbol{z}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{z})^k\right] = \sum_{\boldsymbol{\kappa} \vdash k} \frac{k!\, 2^k}{\prod_{i=1}^k f_i!\, (2i)^{f_i}} \prod_{i=1}^k \left(i\tilde{\theta}_i + \tilde{\tau}_i\right)^{f_i}. \tag{9}$$

Expanding each factor of (9) by the binomial theorem gives the equivalent expression

$$\mathrm{E}\left[(\boldsymbol{z}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{z})^k\right] = \sum_{\boldsymbol{\kappa} \vdash k} \sum_{\boldsymbol{0} \leq \boldsymbol{g} \leq \boldsymbol{f}} c_{\boldsymbol{f},\boldsymbol{g}} \prod_{i=1}^k \tilde{\theta}_i^{g_i}\, \tilde{\tau}_i^{f_i - g_i}, \tag{10}$$

where $\boldsymbol{g} = (g_1, \dots, g_k)$, the inner sum runs over all integer vectors with $0 \leq g_i \leq f_i$ for each i, and

$$c_{\boldsymbol{f},\boldsymbol{g}} = \frac{k!\, 2^k}{\prod_{i=1}^k f_i!\, (2i)^{f_i}} \prod_{i=1}^k \binom{f_i}{g_i} i^{g_i}. \tag{11}$$

Table 1: Number of distinct terms in the expansion (10) of $\mathrm{E}[(z^{\mathsf{T}}\boldsymbol{A}z)^k]$ when $z \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

k	1	2	3	4	5	6	7	8	9	10
$m(k)$	2	5	10	20	36	65	110	185	300	481

No two terms of (10) *coincide, and their total number is*

$$m(k) = \sum_{\boldsymbol{\kappa} \vdash k} \prod_{i=1}^{k} (1 + f_i). \tag{12}$$

Table 1 reports $m(k)$ for $k = 1, \ldots, 10$.

Example 1. For $k = 2$, (10) gives $\mathrm{E}[(z^{\mathsf{T}}\boldsymbol{A}z)^2] = 2\tilde{\tau}_2 + 4\tilde{\theta}_2 + \tilde{\tau}_1^2 + 2\tilde{\theta}_1\tilde{\tau}_1 + \tilde{\theta}_1^2$, which contains $m(2) = 5$ distinct terms.

4. Main result: an expression with no repeated terms

We now turn to the general product

$$\alpha(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k) := \mathrm{E}\left[\prod_{i=1}^{k} z^{\mathsf{T}}\boldsymbol{A}_i z\right]. \tag{13}$$

We first fix some terminology. Let V be a real vector space. A function $\Phi : V^k \to \mathbb{R}$ is *multilinear* if it is linear in each of its k arguments when the remaining $k-1$ arguments are held fixed, and *symmetric* if its value is unchanged by any permutation of its arguments. The *diagonal restriction* of Φ is the single-argument function $P : V \to \mathbb{R}$ defined by $P(\boldsymbol{A}) = \Phi(\boldsymbol{A}, \ldots, \boldsymbol{A})$; the name refers to the fact that P is the restriction of Φ to the diagonal $\{(\boldsymbol{A}, \ldots, \boldsymbol{A}) : \boldsymbol{A} \in V\}$ of V^k.

In our setting, V is the space of symmetric $m \times m$ matrices and $\Phi = \alpha$: since $z^{\mathsf{T}}\boldsymbol{A}z$ is linear in $\boldsymbol{A}$ and the expectation operator is linear, α is multilinear in $(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k)$, and since scalar multiplication commutes, α is symmetric. The diagonal restriction of α is $P(\boldsymbol{A}) = \mathrm{E}[(z^{\mathsf{T}}\boldsymbol{A}z)^k]$, for which Lemma 3 provides an explicit expression. The passage from the one-matrix expression (10) to the k-matrix expression (13) is accomplished by the following polarization identity.

Lemma 4 (Polarization)**.** *Let V be a real vector space and let $\Phi : V^k \to \mathbb{R}$ be a symmetric multilinear function with diagonal restriction $P(\boldsymbol{A}) = \Phi(\boldsymbol{A}, \ldots, \boldsymbol{A})$. Then, for any $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k \in V$,*

$$k!\,\Phi(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k) = [t_1 t_2 \cdots t_k]\, P(t_1\boldsymbol{A}_1 + \cdots + t_k\boldsymbol{A}_k)\,, \tag{14}$$

where $[t_1 \cdots t_k]\, Q(t_1, \ldots, t_k)$ denotes the coefficient of the monomial $t_1 t_2 \cdots t_k$ in the polynomial Q. In particular, a symmetric multilinear function is uniquely determined by its diagonal restriction.

Applying Lemma 4 to α, with P given by (10), produces an expansion of $\alpha(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k)$ into terms of the form (2) and (3) indexed by permutations of $\{1, \ldots, k\}$: each factor $\tilde{\theta}_i^{g_i}\tilde{\tau}_i^{f_i - g_i}$ in (10) opens up into a sum of products of θ's and τ's whose subscripts are words partitioning $\{1, \ldots, k\}$. To write the result down we need notation for these products.

Fix $\boldsymbol{\kappa} \vdash k$ with frequency vector $\boldsymbol{f}$ and let $\boldsymbol{g}$ satisfy $\boldsymbol{0} \le \boldsymbol{g} \le \boldsymbol{f}$. For a permutation $\boldsymbol{\pi}$ of $\{1, \ldots, k\}$, written as a sequence, split $\boldsymbol{\pi}$ into $r = \ell(\boldsymbol{\kappa})$ consecutive blocks $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_r)$ with block lengths in ascending order: the first f_1 blocks are singletons, the next f_2 blocks are pairs, and so on, up to the final f_k blocks of length k (formally, $\ell(\boldsymbol{\pi}_i) = \kappa_{r+1-i}$). Within the group of f_i blocks of length i, designate the first g_i blocks as subscripts of θ-terms and the remaining $f_i - g_i$ blocks as subscripts of τ-terms, and define

$$\eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}} = \prod_{i=1}^{k}\left(\prod_{j=1}^{g_i} \theta_{\boldsymbol{\pi}_{s_i+j}}\right)\left(\prod_{j=g_i+1}^{f_i} \tau_{\boldsymbol{\pi}_{s_i+j}}\right), \qquad s_i = \sum_{l=1}^{i-1} f_l. \tag{15}$$

Example 2. For $\boldsymbol{\kappa} = (2, 2, 1)$ we have $\boldsymbol{f} = (1, 2, 0)$, and every permutation splits as $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_3)$ with $\ell(\boldsymbol{\pi}_1) = 1$ and $\ell(\boldsymbol{\pi}_2) = \ell(\boldsymbol{\pi}_3) = 2$. Taking $\boldsymbol{g} = (0, 1, 0)$ designates $\boldsymbol{\pi}_2$ as a θ-subscript, so $\eta_{\boldsymbol{f},(0,1,0),\boldsymbol{\pi}} = \tau_{\boldsymbol{\pi}_1}\theta_{\boldsymbol{\pi}_2}\tau_{\boldsymbol{\pi}_3}$; taking $\boldsymbol{g} = (1, 0, 0)$ instead gives $\eta_{\boldsymbol{f},(1,0,0),\boldsymbol{\pi}} = \theta_{\boldsymbol{\pi}_1}\tau_{\boldsymbol{\pi}_2}\tau_{\boldsymbol{\pi}_3}$.

By the invariances (4) and the commutativity of scalar multiplication, many permutations $\boldsymbol{\pi}$ produce the same value of $\eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}}$. To eliminate this redundancy we retain only one representative from each class of equivalent permutations.

Definition 1. Fix $\boldsymbol{f}$ and $\boldsymbol{g}$. A permutation $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_r)$, split into blocks as above, is *admissible* if

(i) every block designated as a θ-subscript is a canonical θ-word, and the first elements of the θ-designated blocks of any common length are in ascending order;

(ii) every block designated as a τ-subscript is a canonical τ-word, and the first elements of the τ-designated blocks of any common length are in ascending order.

The set of admissible permutations is denoted $\mathcal{S}_{\boldsymbol{f},\boldsymbol{g}}$.

Since the entries of each block are distinct integers, condition (i) requires the first element of each θ-designated block of length ≥ 2 to be smaller than its last element, while condition (ii) requires the first element of each τ-designated block to be its smallest element and, for blocks of length ≥ 3, the second element to be smaller than the last. The ascending-order requirements remove the redundancy arising from permuting entire blocks of equal length and equal designation.

Example 3. Let $\boldsymbol{\kappa} = (2, 1)$, so $\boldsymbol{f} = (1, 1, 0)$, and let $\boldsymbol{g} = (0, 1, 0)$, which designates the length-two block as a θ-subscript. The admissible permutations are $(1)(2, 3)$, $(2)(1, 3)$, and $(3)(1, 2)$; the permutations $(1)(3, 2)$, $(2)(3, 1)$, and $(3)(2, 1)$ are inadmissible. For $\boldsymbol{\kappa} = (1, 1, 1)$, i.e., $\boldsymbol{f} = (3, 0, 0)$, and $\boldsymbol{g} = (0, 0, 0)$, the only admissible permutation is $(1)(2)(3)$, because the τ-designated singletons must appear in ascending order; for $\boldsymbol{g} = (1, 0, 0)$ there are three admissible permutations, $(1)(2)(3)$, $(2)(1)(3)$, and $(3)(1)(2)$, one for each choice of the θ-designated singleton.

We can now state the main result of the paper.

Proposition 1. *Let $\boldsymbol{z} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}$ positive semidefinite, and let $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ be symmetric matrices. Then*

$$\mathrm{E}\left[\prod_{i=1}^{k} \boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}_i\boldsymbol{z}\right] = \sum_{\boldsymbol{\kappa}\vdash k} \sum_{\boldsymbol{0}\leq\boldsymbol{g}\leq\boldsymbol{f}} 2^{k-f_1-f_2+g_2} \sum_{\boldsymbol{\pi}\in\mathcal{S}_{\boldsymbol{f},\boldsymbol{g}}} \eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}}, \tag{16}$$

with $\eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}}$ as in (15). Moreover:

(i) no two terms on the right-hand side of (16) coincide, i.e., distinct triples $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{\pi})$ with $\boldsymbol{\pi} \in \mathcal{S}_{\boldsymbol{f},\boldsymbol{g}}$ generate distinct products of canonical θ- and τ-terms;

(ii) the number of admissible permutations is

$$h_{\boldsymbol{f},\boldsymbol{g}} = |\mathcal{S}_{\boldsymbol{f},\boldsymbol{g}}| = q(\boldsymbol{\kappa}) \prod_{i=1}^{k} \binom{f_i}{g_i} n_\theta(i)^{g_i}\, n_\tau(i)^{f_i-g_i}, \qquad q(\boldsymbol{\kappa}) = \frac{k!}{\prod_{i=1}^{k}(i!)^{f_i} f_i!}, \tag{17}$$

so the total number of terms in (16) is

$$M(k) = \sum_{\boldsymbol{\kappa}\vdash k} \sum_{\boldsymbol{0}\leq\boldsymbol{g}\leq\boldsymbol{f}} h_{\boldsymbol{f},\boldsymbol{g}} = \sum_{\boldsymbol{\kappa}\vdash k} q(\boldsymbol{\kappa}) \prod_{i=1}^{k} [n_\theta(i) + n_\tau(i)]^{f_i}\,. \tag{18}$$

The coefficient $2^{k-f_1-f_2+g_2}$ has a transparent per-block form: each singleton block contributes a factor 1; each τ-designated pair contributes 2; each θ-designated pair contributes 4; and every block of length $i \geq 3$ contributes 2^i regardless of its designation. This observation is the basis of the efficient re-indexing in Section 5.

Table 2 reports the term counts $M(k)$ for $k = 1, \ldots, 10$. Comparing Tables 1 and 2 shows the price of moving from one matrix to k distinct matrices: for example, $M(10)$ exceeds $m(10)$ by five orders of magnitude.

Table 2: Number of distinct terms in the expansion (16) of $\mathrm{E}[\prod_{i=1}^{k} z^{\mathrm{T}} \boldsymbol{A}_i z]$ when $z \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

k	1	2	3	4	5	6	7	8	9	10
$M(k)$	2	6	24	123	774	5,778	49,824	486,531	5,300,262	63,673,506

Example 4. For $k = 2$, formula (16) yields

$$\mathrm{E}\left[(z^{\mathrm{T}}\boldsymbol{A}_1 z)(z^{\mathrm{T}}\boldsymbol{A}_2 z)\right] = 2\tau_{(1,2)} + 4\theta_{(1,2)} + \tau_1\tau_2 + \theta_1\tau_2 + \theta_2\tau_1 + \theta_1\theta_2,$$

with $M(2) = 6$ distinct terms.

Example 5. For $k = 3$, Table 3 enumerates the admissible permutations for every pair $(\boldsymbol{f}, \boldsymbol{g})$, together with the associated coefficients $2^{k-f_1-f_2+g_2}$ and terms. The number of terms matches $M(3) = 24$ from Table 2. Reading off the table,

$$\begin{aligned}
\mathrm{E}\left[\prod_{i=1}^{3} z^{\mathrm{T}}\boldsymbol{A}_i z\right] = {} & 8\tau_{(1,2,3)} + 8\theta_{(1,2,3)} + 8\theta_{(1,3,2)} + 8\theta_{(2,1,3)} \\
& + 2\tau_1\tau_{(2,3)} + 2\tau_2\tau_{(1,3)} + 2\tau_3\tau_{(1,2)} + 4\tau_1\theta_{(2,3)} + 4\tau_2\theta_{(1,3)} + 4\tau_3\theta_{(1,2)} \\
& + 2\theta_1\tau_{(2,3)} + 2\theta_2\tau_{(1,3)} + 2\theta_3\tau_{(1,2)} + 4\theta_1\theta_{(2,3)} + 4\theta_2\theta_{(1,3)} + 4\theta_3\theta_{(1,2)} \\
& + \tau_1\tau_2\tau_3 + \theta_1\tau_2\tau_3 + \theta_2\tau_1\tau_3 + \theta_3\tau_1\tau_2 + \theta_1\theta_2\tau_3 + \theta_1\theta_3\tau_2 + \theta_2\theta_3\tau_1 + \theta_1\theta_2\theta_3. \qquad (19)
\end{aligned}$$

When $\boldsymbol{\mu} = \boldsymbol{0}_m$, all θ-terms vanish, so only the terms with $\boldsymbol{g} = \boldsymbol{0}$ survive in (16), and we recover the classical result of Don [3].

Corollary 1. *Let $z \sim \mathcal{N}(\boldsymbol{0}_m, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}$ positive semidefinite, and let $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ be symmetric matrices. Then*

$$\mathrm{E}\left[\prod_{i=1}^{k} z^{\mathrm{T}}\boldsymbol{A}_i z\right] = \sum_{\boldsymbol{\kappa} \vdash k} 2^{k-f_1-f_2} \sum_{\boldsymbol{\pi} \in \mathcal{S}_{\boldsymbol{\kappa}}} \tau_{\pi_1} \cdots \tau_{\pi_r}, \qquad (20)$$

where $\mathcal{S}_{\boldsymbol{\kappa}} = \mathcal{S}_{\boldsymbol{f},\boldsymbol{0}}$ is the set of permutations whose blocks are all canonical τ-words with first elements of equal-length blocks in ascending order. The number of terms in (20) corresponding to a given $\boldsymbol{\kappa}$ is

$$\varphi(\boldsymbol{\kappa}) = h_{\boldsymbol{f},\boldsymbol{0}} = \frac{k!\, 2^{f_1+f_2}}{\prod_{i=1}^{k} f_i!\,(2i)^{f_i}}, \qquad (21)$$

and the total number of terms is $N(k) = \sum_{\boldsymbol{\kappa} \vdash k} \varphi(\boldsymbol{\kappa})$.

Table 4 reports $N(k)$ for $k = 1, \ldots, 10$. The sequence $N(k)$ also counts the ways to cover k labeled vertices by disjoint undirected cycles (entry A002135 of the On-Line Encyclopedia of Integer Sequences [21]), and satisfies the recurrence

$$N(k+1) = (k+1)N(k) - \binom{k}{2} N(k-2),$$

with $N(0) = N(1) = 1$ and $N(2) = 2$.

5. A computationally efficient expression

Although (16) and (20) are explicit and free of repetitions, enumerating their terms directly is expensive: for every pair $(\boldsymbol{f}, \boldsymbol{g})$ one must screen all $k!$ permutations of $\{1, \ldots, k\}$ for admissibility. In this section we re-index the same expression by set partitions, which removes the screening step entirely.

The idea is contained in the per-block factorization of the coefficient $2^{k-f_1-f_2+g_2}$ noted after Proposition 1: the coefficient, the admissibility conditions, and the term $\eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}}$ all factor across the blocks of $\boldsymbol{\pi}$. Consequently, instead of first

Table 3: Construction of the explicit expression of $\mathrm{E}[\prod_{i=1}^{3} z^{\mathrm{T}} \boldsymbol{A}_i z]$ for $z \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ using (16).

$\boldsymbol{g}$	$2^{k-f_1-f_2+g_2}$	Admissible $\boldsymbol{\pi}$	Term
	$\boldsymbol{\kappa} = (3), \boldsymbol{f} = (0,0,1)$		
(0,0,0)	8	(1,2,3)	$\tau_{(1,2,3)}$
(0,0,1)	8	(1,2,3)	$\theta_{(1,2,3)}$
		(1,3,2)	$\theta_{(1,3,2)}$
		(2,1,3)	$\theta_{(2,1,3)}$
	$\boldsymbol{\kappa} = (2,1), \boldsymbol{f} = (1,1,0)$		
(0,0,0)	2	(1)(2,3)	$\tau_1\tau_{(2,3)}$
		(2)(1,3)	$\tau_2\tau_{(1,3)}$
		(3)(1,2)	$\tau_3\tau_{(1,2)}$
(0,1,0)	4	(1)(2,3)	$\tau_1\theta_{(2,3)}$
		(2)(1,3)	$\tau_2\theta_{(1,3)}$
		(3)(1,2)	$\tau_3\theta_{(1,2)}$
(1,0,0)	2	(1)(2,3)	$\theta_1\tau_{(2,3)}$
		(2)(1,3)	$\theta_2\tau_{(1,3)}$
		(3)(1,2)	$\theta_3\tau_{(1,2)}$
(1,1,0)	4	(1)(2,3)	$\theta_1\theta_{(2,3)}$
		(2)(1,3)	$\theta_2\theta_{(1,3)}$
		(3)(1,2)	$\theta_3\theta_{(1,2)}$
	$\boldsymbol{\kappa} = (1,1,1), \boldsymbol{f} = (3,0,0)$		
(0,0,0)	1	(1)(2)(3)	$\tau_1\tau_2\tau_3$
(1,0,0)	1	(1)(2)(3)	$\theta_1\tau_2\tau_3$
		(2)(1)(3)	$\theta_2\tau_1\tau_3$
		(3)(1)(2)	$\theta_3\tau_1\tau_2$
(2,0,0)	1	(1)(2)(3)	$\theta_1\theta_2\tau_3$
		(1)(3)(2)	$\theta_1\theta_3\tau_2$
		(2)(3)(1)	$\theta_2\theta_3\tau_1$
(3,0,0)	1	(1)(2)(3)	$\theta_1\theta_2\theta_3$

fixing the designation pattern $(\boldsymbol{f}, \boldsymbol{g})$ and then enumerating admissible permutations, we may first fix an (unordered) set partition of $\{1, \ldots, k\}$ and then sum over all ways to convert each block into a canonical θ- or τ-term.

For a word $\boldsymbol{\nu}$ with i distinct entries, denote by $\boldsymbol{p}_j(\boldsymbol{\nu})$, $j = 1, \ldots, n_\theta(i)$, the canonical θ-words formed from the entries of $\boldsymbol{\nu}$, and by $\boldsymbol{q}_j(\boldsymbol{\nu})$, $j = 1, \ldots, n_\tau(i)$, the canonical τ-words (bracelets) formed from the entries of $\boldsymbol{\nu}$.[1] Define the *block sum*

$$\lambda_{\boldsymbol{\nu}} = 2^{\mathbf{1}\{\ell(\boldsymbol{\nu})=2\}} \sum_{j=1}^{n_\theta(\ell(\boldsymbol{\nu}))} \theta_{\boldsymbol{p}_j(\boldsymbol{\nu})} + \sum_{j=1}^{n_\tau(\ell(\boldsymbol{\nu}))} \tau_{\boldsymbol{q}_j(\boldsymbol{\nu})}, \tag{22}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function. The factor $2^{\mathbf{1}\{\ell(\boldsymbol{\nu})=2\}}$ attached to the θ-terms reproduces the extra 2^{g_2} in the coefficient of (16).

Example 6. For $\boldsymbol{\nu} = (3,7,9)$ we have $n_\theta(3) = 3$ canonical θ-words, namely $(3,7,9)$, $(3,9,7)$, and $(7,3,9)$, and $n_\tau(3) = 1$ canonical τ-word, namely $(3,7,9)$. Hence $\lambda_{(3,7,9)} = \theta_{(3,7,9)} + \theta_{(3,9,7)} + \theta_{(7,3,9)} + \tau_{(3,7,9)}$.

[1] An efficient algorithm for generating bracelets with fixed content is given by Karim, Sawada, Alamgir, and Husnine [11]; it is easily adapted to generate canonical θ-words as well.

Table 4: Number of distinct terms in the expansion (20) of $\mathrm{E}[\prod_{i=1}^{k} z^{\mathrm{T}} A_i z]$ when $z \sim \mathcal{N}(\mathbf{0}_m, \boldsymbol{\Sigma})$.

k	1	2	3	4	5	6	7	8	9	10
$N(k)$	1	2	5	17	73	388	2,461	18,155	152,531	1,436,714

Table 5: Construction of the explicit expression of $\mathrm{E}[\prod_{i=1}^{3} z^{\mathrm{T}} A_i z]$ for $z \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ using (23).

$\boldsymbol{\pi} \in \mathcal{B}_3$	$2^{k-f_1-f_2}$	λ_{π_1}	λ_{π_2}	λ_{π_3}
(1)(2)(3)	1	$\theta_1 + \tau_1$	$\theta_2 + \tau_2$	$\theta_3 + \tau_3$
(1)(2,3)	2	$\theta_1 + \tau_1$	$2\theta_{(2,3)} + \tau_{(2,3)}$	–
(2)(1,3)	2	$\theta_2 + \tau_2$	$2\theta_{(1,3)} + \tau_{(1,3)}$	–
(3)(1,2)	2	$\theta_3 + \tau_3$	$2\theta_{(1,2)} + \tau_{(1,2)}$	–
(1,2,3)	8	$\theta_{(1,2,3)} + \theta_{(1,3,2)} + \theta_{(2,1,3)} + \tau_{(1,2,3)}$	–	–

Proposition 2. *Let $z \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}$ positive semidefinite, and let $A_1, \dots, A_k$ be symmetric matrices. Then*

$$\mathrm{E}\left[\prod_{i=1}^{k} z^{\mathrm{T}} A_i z\right] = \sum_{\boldsymbol{\pi} \in \mathcal{B}_k} 2^{k-f_1-f_2} \prod_{i=1}^{\ell(\boldsymbol{\pi})} \lambda_{\pi_i}, \tag{23}$$

where f_1 and f_2 denote the number of blocks of $\boldsymbol{\pi}$ with one and two elements, respectively. Expanding the products in (23) yields exactly the $M(k)$ distinct terms of (16), each with the same coefficient. When $\boldsymbol{\mu} = \mathbf{0}_m$, the θ-terms in (22) vanish and (23) yields the $N(k)$ distinct terms of (20).

The computational advantage of (23) over (16) is substantial. The sum in (23) runs over the B_k set partitions of $\{1, \dots, k\}$, which can be generated directly [20], and the canonical words within each block can also be generated directly [11]; no permutation ever needs to be examined and rejected. Since B_k is far smaller than $k!$ (for instance, $B_{10} = 115{,}975$ while $10! = 3{,}628{,}800$), the gain grows rapidly with k.

Example 7. We revisit the case $k = 3$ using (23). Table 5 lists the five set partitions of $\{1, 2, 3\}$ and the corresponding block sums, from which

$$\begin{aligned}\mathrm{E}\left[\prod_{i=1}^{3} z^{\mathrm{T}} A_i z\right] = {} & (\theta_1 + \tau_1)(\theta_2 + \tau_2)(\theta_3 + \tau_3) + 2(\theta_1 + \tau_1)\left[2\theta_{(2,3)} + \tau_{(2,3)}\right] \\ & + 2(\theta_2 + \tau_2)\left[2\theta_{(1,3)} + \tau_{(1,3)}\right] + 2(\theta_3 + \tau_3)\left[2\theta_{(1,2)} + \tau_{(1,2)}\right] \\ & + 8\left[\theta_{(1,2,3)} + \theta_{(1,3,2)} + \theta_{(2,1,3)} + \tau_{(1,2,3)}\right].\end{aligned}$$

Expanding this expression reproduces (19) term by term.

6. Hermitian quadratic forms in complex normal random vectors

We now extend the preceding results to complex normal random vectors. The multivariate complex normal distribution goes back to Wooding [26]. Let $z = z_r + \mathrm{i} z_c$ be a complex random m-vector, where $z_r = \Re(z)$, $z_c = \Im(z)$, and $\mathrm{i} = \sqrt{-1}$. Let $\boldsymbol{\mu} = \boldsymbol{\mu}_r + \mathrm{i}\boldsymbol{\mu}_c$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_r + \mathrm{i}\boldsymbol{\Sigma}_c$, where $\boldsymbol{\Sigma}_r$ is real symmetric and $\boldsymbol{\Sigma}_c$ is real skew-symmetric, and define the *augmented real representations*

$$\hat{\boldsymbol{\mu}} = \begin{bmatrix} \boldsymbol{\mu}_r \\ \boldsymbol{\mu}_c \end{bmatrix}, \qquad \hat{\boldsymbol{\Sigma}} = \frac{1}{2}\begin{bmatrix} \boldsymbol{\Sigma}_r & -\boldsymbol{\Sigma}_c \\ \boldsymbol{\Sigma}_c & \boldsymbol{\Sigma}_r \end{bmatrix}, \tag{24}$$

and assume that the symmetric matrix $\hat{\boldsymbol{\Sigma}}$ is positive semidefinite. If $\hat{z} = (z_r^{\mathrm{T}}, z_c^{\mathrm{T}})^{\mathrm{T}} \sim \mathcal{N}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}})$, we say that z follows a (circularly symmetric) multivariate complex normal distribution, written $z \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

Let $\boldsymbol{A} = \boldsymbol{A}_r + \mathrm{i}\boldsymbol{A}_c$ be a Hermitian matrix, so that $\boldsymbol{A}_r$ is real symmetric and $\boldsymbol{A}_c$ is real skew-symmetric, and let $\boldsymbol{z}^{\mathrm{H}}\boldsymbol{A}\boldsymbol{z}$ denote the associated Hermitian quadratic form, where H denotes the conjugate transpose. Setting

$$\hat{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A}_r & -\boldsymbol{A}_c \\ \boldsymbol{A}_c & \boldsymbol{A}_r \end{bmatrix}, \tag{25}$$

a direct computation shows that $\boldsymbol{z}^{\mathrm{H}}\boldsymbol{A}\boldsymbol{z}$ is real and equals $\hat{\boldsymbol{z}}^{\mathrm{T}}\hat{\boldsymbol{A}}\hat{\boldsymbol{z}}$. Consequently, for Hermitian matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$,

$$\mathrm{E}\left[\prod_{i=1}^{k} \boldsymbol{z}^{\mathrm{H}}\boldsymbol{A}_i\boldsymbol{z}\right] = \mathrm{E}\left[\prod_{i=1}^{k} \hat{\boldsymbol{z}}^{\mathrm{T}}\hat{\boldsymbol{A}}_i\hat{\boldsymbol{z}}\right] \tag{26}$$

is the expectation of a product of quadratic forms in a real normal vector, and Propositions 1 and 2 apply verbatim to the augmented quantities. The next lemma shows that the augmented θ- and τ-terms reduce to real parts of the corresponding complex expressions, so that the augmented representation never needs to be formed explicitly.

Lemma 5. *Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_p)$ be a word and define*

$$\theta_{\boldsymbol{\nu}} = \Re\left(\boldsymbol{\mu}^{\mathrm{H}}\boldsymbol{A}_{\nu_1}\boldsymbol{\Sigma}\boldsymbol{A}_{\nu_2}\boldsymbol{\Sigma}\cdots\boldsymbol{\Sigma}\boldsymbol{A}_{\nu_p}\boldsymbol{\mu}\right), \tag{27}$$

$$\tau_{\boldsymbol{\nu}} = \Re\left(\mathrm{tr}\left(\boldsymbol{A}_{\nu_1}\boldsymbol{\Sigma}\boldsymbol{A}_{\nu_2}\boldsymbol{\Sigma}\cdots\boldsymbol{A}_{\nu_p}\boldsymbol{\Sigma}\right)\right). \tag{28}$$

Then the corresponding quantities formed from $(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{A}}_1, \ldots, \hat{\boldsymbol{A}}_k)$ satisfy

$$\hat{\theta}_{\boldsymbol{\nu}} = \frac{\theta_{\boldsymbol{\nu}}}{2^{p-1}}, \qquad \hat{\tau}_{\boldsymbol{\nu}} = \frac{\tau_{\boldsymbol{\nu}}}{2^{p-1}}. \tag{29}$$

Substituting (29) into Propositions 1 and 2 and collecting the powers of two yields the complex analogue of our main results.

Proposition 3. *Let $\boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and let $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ be Hermitian matrices. With $\theta_{\boldsymbol{\nu}}$ and $\tau_{\boldsymbol{\nu}}$ defined by (27) and (28),*

$$\mathrm{E}\left[\prod_{i=1}^{k} \boldsymbol{z}^{\mathrm{H}}\boldsymbol{A}_i\boldsymbol{z}\right] = \sum_{\boldsymbol{\kappa}\vdash k}\sum_{\boldsymbol{0}\le\boldsymbol{g}\le\boldsymbol{f}} 2^{\ell(\boldsymbol{\kappa})-f_1-f_2+g_2} \sum_{\boldsymbol{\pi}\in\mathcal{S}_{\boldsymbol{f},\boldsymbol{g}}} \eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}} \tag{30}$$

and, equivalently,

$$\mathrm{E}\left[\prod_{i=1}^{k} \boldsymbol{z}^{\mathrm{H}}\boldsymbol{A}_i\boldsymbol{z}\right] = \sum_{\boldsymbol{\pi}\in\mathcal{B}_k} 2^{\ell(\boldsymbol{\pi})-f_1-f_2} \prod_{i=1}^{\ell(\boldsymbol{\pi})} \lambda_{\boldsymbol{\pi}_i}, \tag{31}$$

with $\eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}}$ and $\lambda_{\boldsymbol{\pi}_i}$ as in (15) and (22).

Comparing Proposition 3 with Propositions 1 and 2, the complex case requires two modifications: the coefficient $2^{k-f_1-f_2+g_2}$ becomes $2^{\ell(\boldsymbol{\kappa})-f_1-f_2+g_2}$ (equivalently, $2^{k-f_1-f_2}$ becomes $2^{\ell(\boldsymbol{\pi})-f_1-f_2}$), and the real part must be taken in the definitions of $\theta_{\boldsymbol{\nu}}$ and $\tau_{\boldsymbol{\nu}}$.

The second modification is easy to overlook but essential. For words of length one or two the trace terms are automatically real, since, e.g., $\overline{\mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma})} = \mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma}\boldsymbol{A}_1) = \mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma})$ by conjugate transposition and cyclic invariance. For longer words, however, the imaginary part is generally nonzero.[2] For example, Sultan and Tracy [25, Eq. (5.3)] state that, for $\boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{0}_m, \boldsymbol{\Sigma})$,

$$\begin{aligned}\mathrm{E}\left[\prod_{i=1}^{3} \boldsymbol{z}^{\mathrm{H}}\boldsymbol{A}_i\boldsymbol{z}\right] &= 2\,\mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma}\boldsymbol{A}_3\boldsymbol{\Sigma}) + \mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma})\,\mathrm{tr}(\boldsymbol{A}_2\boldsymbol{\Sigma}\boldsymbol{A}_3\boldsymbol{\Sigma}) + \mathrm{tr}(\boldsymbol{A}_2\boldsymbol{\Sigma})\,\mathrm{tr}(\boldsymbol{A}_3\boldsymbol{\Sigma}\boldsymbol{A}_1\boldsymbol{\Sigma}) \\ &\quad + \mathrm{tr}(\boldsymbol{A}_3\boldsymbol{\Sigma})\,\mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma}) + \mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma})\,\mathrm{tr}(\boldsymbol{A}_2\boldsymbol{\Sigma})\,\mathrm{tr}(\boldsymbol{A}_3\boldsymbol{\Sigma}),\end{aligned} \tag{32}$$

[2] Similarly, in the noncentral case, $\boldsymbol{\mu}^{\mathrm{H}}\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\mu}$ has a nonzero imaginary part in general, so the real part must be taken even for words of length two in (27).

whereas our (31) gives the same expression with $2\,\mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma}\boldsymbol{A}_3\boldsymbol{\Sigma})$ replaced by $2\Re\,(\mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma}\boldsymbol{A}_3\boldsymbol{\Sigma}))$.[3] Since $\Im(\mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma}\boldsymbol{A}_3\boldsymbol{\Sigma})) \neq 0$ in general, (32) is not real-valued and hence cannot equal the left-hand side. Taking the real part of (32) does repair it for $k = 3$, but this stopgap fails for larger products: when $k \geq 6$ in the central case, or $k \geq 4$ in the noncentral case, even the real part of the expression is incorrect unless the real part is taken term by term as in (27) and (28).

An alternative correct expression for the central complex case follows from Theorem 1 of Graczyk, Letac, and Massam [4] with $p = 1$ in their notation:

$$\begin{aligned}\mathrm{E}\left[\prod_{i=1}^{3} z^{\mathrm{H}}\boldsymbol{A}_i z\right] &= \mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma}\boldsymbol{A}_3\boldsymbol{\Sigma}) + \mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_3\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma}) + \mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma})\,\mathrm{tr}(\boldsymbol{A}_2\boldsymbol{\Sigma}\boldsymbol{A}_3\boldsymbol{\Sigma}) \\ &\quad + \mathrm{tr}(\boldsymbol{A}_2\boldsymbol{\Sigma})\,\mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_3\boldsymbol{\Sigma}) + \mathrm{tr}(\boldsymbol{A}_3\boldsymbol{\Sigma})\,\mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma}) + \mathrm{tr}(\boldsymbol{A}_1\boldsymbol{\Sigma})\,\mathrm{tr}(\boldsymbol{A}_2\boldsymbol{\Sigma})\,\mathrm{tr}(\boldsymbol{A}_3\boldsymbol{\Sigma}).\end{aligned} \tag{33}$$

Here the imaginary parts of the first two terms cancel, so (33) is correct without any explicit real-part operation. The price is redundancy: for general k, the formula of [4] contains $k!$ terms, whereas (31) contains only $N(k)$ terms in the central case, and Table 4 shows that $N(k)$ is dramatically smaller than $k!$ for large k.

7. Product moments of Wishart distributions

The results of the previous sections deliver, almost for free, explicit expressions for the product moments of central and noncentral, real and complex Wishart distributions.

Let $z_t \sim \mathcal{N}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma})$, $t = 1, \ldots, n$, be independent. Then

$$\boldsymbol{W} = \sum_{t=1}^{n} z_t z_t^{\mathrm{T}} \sim \mathcal{W}_m(n, \boldsymbol{\Sigma}, \boldsymbol{\Lambda})$$

follows a noncentral real Wishart distribution with n degrees of freedom, covariance matrix $\boldsymbol{\Sigma}$, and noncentrality matrix $\boldsymbol{\Lambda} = \sum_{t=1}^{n} \boldsymbol{\mu}_t\boldsymbol{\mu}_t^{\mathrm{T}}$. (The conventional noncentrality parameter is $\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\Lambda}$; we parameterize by the pair $(\boldsymbol{\Sigma}, \boldsymbol{\Lambda})$ so as to accommodate singular $\boldsymbol{\Sigma}$.) When all $\boldsymbol{\mu}_t = \boldsymbol{0}_m$, the distribution is the central real Wishart. Analogously, if $z_t \sim \mathcal{CN}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma})$, $t = 1, \ldots, n$, are independent, then $\boldsymbol{W} = \sum_{t=1}^{n} z_t z_t^{\mathrm{H}} \sim \mathcal{CW}_m(n, \boldsymbol{\Sigma}, \boldsymbol{\Lambda})$ follows a noncentral complex Wishart distribution with noncentrality matrix $\boldsymbol{\Lambda} = \sum_{t=1}^{n} \boldsymbol{\mu}_t\boldsymbol{\mu}_t^{\mathrm{H}}$.

Our goal is an explicit expression for

$$\mathrm{E}\left[\prod_{i=1}^{k} \mathrm{tr}(\boldsymbol{W}\boldsymbol{A}_i)\right], \tag{34}$$

where the $\boldsymbol{A}_i$ are symmetric in the real case and Hermitian in the complex case. Expression (34) contains the product moments of the individual entries of $\boldsymbol{W}$ as special cases: taking $\boldsymbol{A} = (\boldsymbol{e}_i\boldsymbol{e}_j^{\mathrm{T}} + \boldsymbol{e}_j\boldsymbol{e}_i^{\mathrm{T}})/2$, where $\boldsymbol{e}_i$ is the i-th column of $\mathbf{I}_m$, gives $\mathrm{tr}(\boldsymbol{W}\boldsymbol{A}) = w_{ij}$, the (i, j)-th entry of $\boldsymbol{W}$.

The key observation is that each $\mathrm{tr}(\boldsymbol{W}\boldsymbol{A}_i)$ is itself a quadratic form in a normal vector of dimension mn, so (34) is covered by Propositions 1–3; the Kronecker structure collapses the resulting θ- and τ-terms to $m \times m$ computations.

Corollary 2. *Let $\boldsymbol{W} \sim \mathcal{W}_m(n, \boldsymbol{\Sigma}, \boldsymbol{\Lambda})$ with $\boldsymbol{\Sigma}$ positive semidefinite, and let $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ be symmetric matrices. Then $\mathrm{E}[\prod_{i=1}^{k} \mathrm{tr}(\boldsymbol{W}\boldsymbol{A}_i)]$ is given by (16) and (23), with the θ- and τ-terms replaced by*

$$\theta_{\boldsymbol{\nu}} = \mathrm{tr}\left(\boldsymbol{A}_{\nu_1}\boldsymbol{\Sigma}\boldsymbol{A}_{\nu_2}\boldsymbol{\Sigma}\cdots\boldsymbol{\Sigma}\boldsymbol{A}_{\nu_p}\boldsymbol{\Lambda}\right), \tag{35}$$

$$\tau_{\boldsymbol{\nu}} = n\,\mathrm{tr}\left(\boldsymbol{A}_{\nu_1}\boldsymbol{\Sigma}\boldsymbol{A}_{\nu_2}\boldsymbol{\Sigma}\cdots\boldsymbol{A}_{\nu_p}\boldsymbol{\Sigma}\right). \tag{36}$$

Corollary 3. *Let $\boldsymbol{W} \sim \mathcal{CW}_m(n, \boldsymbol{\Sigma}, \boldsymbol{\Lambda})$ and let $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ be Hermitian matrices. Then $\mathrm{E}[\prod_{i=1}^{k} \mathrm{tr}(\boldsymbol{W}\boldsymbol{A}_i)]$ is given by (30) and (31), with the θ- and τ-terms replaced by the real parts of (35) and (36).*

[3] Their Eq. (5.2) for $\mathrm{E}[(z^{\mathrm{H}}\boldsymbol{A}_1 z)(z^{\mathrm{H}}\boldsymbol{A}_2 z)]$ also contains an apparent typographical error: the coefficient of the first term should be 1 rather than 2.

For the central case ($\mathbf{\Lambda} = \mathbf{0}_{m\times m}$), explicit formulas for (34) were given by Graczyk, Letac, and Massam, in [4] for the complex case and in [5] for the real case; these formulas contain $k!$ terms, whereas ours contain $N(k)$ terms. For the noncentral case, Letac and Massam [14] describe a method for obtaining (34) and work out the cases $k = 1, 2, 3$.[4] However, the expression they report for $k = 3$ (p. 1405) is missing six terms, which illustrates how difficult it is to obtain correct explicit expressions by hand; a counting argument shows that their method produces $\sum_{i=0}^{k} i!\binom{k}{i}^2$ terms, i.e., 34 terms for $k = 3$ and 209 terms for $k = 4$, compared with $M(3) = 24$ and $M(4) = 123$ terms in our expression. Di Nardo [2] proposes a symbolic method for the complex noncentral case together with a Maple implementation; unfortunately, the expression reported in her Example 3 for $\mathrm{E}[\mathrm{tr}(\boldsymbol{W}\boldsymbol{A}_1)\,\mathrm{tr}(\boldsymbol{W}\boldsymbol{A}_2)^2]$ is also incorrect. We return to this example in Section 8 (Example 12), where the correct expression is derived from our results.

8. Products with repeated quadratic forms

In this final section we derive an explicit expression for

$$\mathrm{E}\left[\prod_{i=1}^{k} (\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}_i\boldsymbol{z})^{s_i}\right], \tag{37}$$

where $s_1, \ldots, s_k$ are positive integers. Of course, (37) is the special case of (1) in which the list of matrices contains repetitions, so Propositions 1 and 2 apply after expanding the list. Repetitions, however, make many terms of (23) coincide, and the number of genuinely distinct terms can be smaller by orders of magnitude. The expression derived below (Proposition 4) enumerates each distinct term exactly once, with its multiplicity given in closed form.

8.1. Multiset partitions

Let $\boldsymbol{s} = (s_1, \ldots, s_k)$ and $|\boldsymbol{s}| = \sum_{i=1}^{k} s_i$, and consider the multiset containing s_i copies of the integer i for $i = 1, \ldots, k$. We write $\mathcal{B}_{\boldsymbol{s}}$ for the collection of partitions of this multiset into nonempty blocks (multisets), listed, as before, with no significance attached to the order of the blocks. For example, for $\boldsymbol{s} = (2, 2)$, the multiset $\{1, 1, 2, 2\}$ has the nine partitions

$$(1)(1)(2)(2),\quad (1)(1)(2,2),\quad (1)(2)(1,2),\quad (2)(2)(1,1),\quad (1,1)(2,2),\quad (1,2)(1,2),\quad (1)(1,2,2),\quad (2)(1,1,2),\quad (1,1,2,2).$$

Explicit and recursive formulas for $|\mathcal{B}_{\boldsymbol{s}}|$ can be found in MacMahon [15] and Gupta [6], respectively, and Knuth [12, Algorithm M, pp. 429–430] provides an efficient algorithm for enumerating $\mathcal{B}_{\boldsymbol{s}}$. When $s_1 = \cdots = s_k = 1$, $\mathcal{B}_{\boldsymbol{s}}$ reduces to $\mathcal{B}_k$.

Unlike a set partition, a multiset partition $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_r) \in \mathcal{B}_{\boldsymbol{s}}$ may contain repeated blocks (e.g., $(1,2)(1,2)$ above). We write $\tilde{\boldsymbol{\pi}}_1, \ldots, \tilde{\boldsymbol{\pi}}_{r'}$ for the distinct blocks of $\boldsymbol{\pi}$ and $h_1, \ldots, h_{r'}$ for their multiplicities, and we let b_{ij} denote the number of copies of the integer j in block $\boldsymbol{\pi}_i$.

Our strategy is to start from Proposition 2 applied to the expanded list of $|\boldsymbol{s}|$ matrices

$$\tilde{\boldsymbol{A}}_1 = \cdots = \tilde{\boldsymbol{A}}_{s_1} = \boldsymbol{A}_1, \quad \tilde{\boldsymbol{A}}_{s_1+1} = \cdots = \tilde{\boldsymbol{A}}_{s_1+s_2} = \boldsymbol{A}_2, \quad \ldots, \quad \tilde{\boldsymbol{A}}_{|\boldsymbol{s}|-s_k+1} = \cdots = \tilde{\boldsymbol{A}}_{|\boldsymbol{s}|} = \boldsymbol{A}_k, \tag{38}$$

and to group the set partitions $\boldsymbol{\pi}' \in \mathcal{B}_{|\boldsymbol{s}|}$ that become indistinguishable after the substitution (38). Each $\boldsymbol{\pi}' \in \mathcal{B}_{|\boldsymbol{s}|}$ collapses to the multiset partition $\boldsymbol{\pi} \in \mathcal{B}_{\boldsymbol{s}}$ obtained by replacing every element of every block by the index of the matrix it represents. The first ingredient we need is the number of $\boldsymbol{\pi}'$ that collapse to a given $\boldsymbol{\pi}$.

Lemma 6. *For $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_r) \in \mathcal{B}_{\boldsymbol{s}}$, the number of set partitions $\boldsymbol{\pi}' \in \mathcal{B}_{|\boldsymbol{s}|}$ that collapse to $\boldsymbol{\pi}$ under the substitution* (38) *is*

$$c_{\boldsymbol{s}}(\boldsymbol{\pi}) = \frac{1}{\prod_{i=1}^{r'} h_i!} \prod_{j=1}^{k} \frac{s_j!}{\prod_{i=1}^{r} b_{ij}!}. \tag{39}$$

[4]Although [14] treats the real Wishart case, the method applies equally to the complex case: in their notation, the real case corresponds to $p = n/2$ and $\sigma = 2\mathbf{\Sigma}$, and the complex case to $p = n$ and $\sigma = \mathbf{\Sigma}$.

Table 6: Collapse of the set partitions in $\mathcal{B}_4$ to the multiset partitions in $\mathcal{B}_{(2,2)}$.

$\boldsymbol{\pi} \in \mathcal{B}_{(2,2)}$	$\boldsymbol{\pi}' \in \mathcal{B}_4$
(1)(1)(2)(2)	(1)(2)(3)(4)
(1)(1)(2,2)	(1)(2)(3,4)
(1)(2)(1,2)	(1)(3)(2,4), (1)(4)(2,3), (2)(3)(1,4), (2)(4)(1,3)
(2)(2)(1,1)	(3)(4)(1,2)
(1,1)(2,2)	(1,2)(3,4)
(1,2)(1,2)	(1,3)(2,4), (1,4)(2,3)
(1)(1,2,2)	(1)(2,3,4), (2)(1,3,4)
(2)(1,1,2)	(3)(1,2,4), (4)(1,2,3)
(1,1,2,2)	(1,2,3,4)

Example 8. Let $\boldsymbol{s} = (2, 2)$. Table 6 displays the fifteen set partitions of $\mathcal{B}_4$ grouped by the multiset partitions of $\mathcal{B}_{(2,2)}$ to which they collapse (here $\tilde{\boldsymbol{A}}_1 = \tilde{\boldsymbol{A}}_2 = \boldsymbol{A}_1$ and $\tilde{\boldsymbol{A}}_3 = \tilde{\boldsymbol{A}}_4 = \boldsymbol{A}_2$). For $\boldsymbol{\pi} = (1)(1)(2)(2)$ there are two distinct blocks, each of multiplicity two, and all $b_{ij} \in \{0, 1\}$, so $c_{\boldsymbol{s}}(\boldsymbol{\pi}) = (2!\,2!)/(2!\,2!) = 1$. For $\boldsymbol{\pi} = (1)(2)(1, 2)$ the three blocks are distinct and all $b_{ij} \in \{0, 1\}$, so $c_{\boldsymbol{s}}(\boldsymbol{\pi}) = 2!\,2! = 4$, in agreement with the four set partitions listed in the third group of Table 6.

8.2. *Counting distinct terms*

The second ingredient is a count of the distinct θ- and τ-terms that can be formed from a block that contains repeated indices. Let $\boldsymbol{\nu}$ be a word whose distinct entries have multiplicities $n_1, \ldots, n_p$, and let $n = \sum_{i=1}^{p} n_i = \ell(\boldsymbol{\nu})$. The number of distinct multiset permutations of the entries of $\boldsymbol{\nu}$ is the multinomial coefficient

$$N(n_1, \ldots, n_p) = \frac{n!}{\prod_{i=1}^{p} n_i!}, \tag{40}$$

but when multiplicities are present, fewer distinct θ- and τ-terms can be formed, because distinct arrangements may coincide after the identifications (4). We write $n_\theta(\boldsymbol{\nu})$ and $n_\tau(\boldsymbol{\nu})$ for the number of distinct θ- and τ-terms, generalizing $n_\theta(i)$ and $n_\tau(i)$ of Lemma 1.

Lemma 7. *Let $\boldsymbol{\nu}$ be a word of length n whose distinct entries have multiplicities $n_1, \ldots, n_p$. Then*

$$n_\theta(\boldsymbol{\nu}) = \frac{N(n_1, \ldots, n_p) + P(n_1, \ldots, n_p)}{2}, \tag{41}$$

where $P(n_1, \ldots, n_p)$ is the number of palindromic arrangements (arrangements equal to their own reversal), given by

$$P(n_1, \ldots, n_p) = \begin{cases} \dfrac{\lfloor n/2 \rfloor!}{\prod_{i=1}^{p} \lfloor n_i/2 \rfloor!}, & \text{if at most one } n_i \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases} \tag{42}$$

Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Example 9. For $\boldsymbol{\nu} = (1, 1, 2, 2)$ we have $N = 6$ and $P = 2$, so $n_\theta(\boldsymbol{\nu}) = 4$: the arrangements $(1, 1, 2, 2)$, $(1, 2, 1, 2)$, $(1, 2, 2, 1)$, and $(2, 1, 1, 2)$ are canonical θ-words, while $(2, 1, 2, 1)$ and $(2, 2, 1, 1)$ exceed their reversals lexicographically.

By (4), $n_\tau(\boldsymbol{\nu})$ equals the number of *bracelets with fixed content* $(n_1, \ldots, n_p)$. The number of *necklaces* with fixed content is classical, and follows from Pólya's enumeration theorem [22] applied to the cyclic group (see, e.g., Read [23]):

$$N_{\text{neck}}(n_1, \ldots, n_p) = \frac{1}{n} \sum_{j \,|\, \gcd(n_1, \ldots, n_p)} \phi(j) \frac{(n/j)!}{\prod_{i=1}^{p} (n_i/j)!}, \tag{43}$$

Table 7: Number of distinct terms in the expansion of $\mathrm{E}[\prod_{i=1}^{k}(z^{\mathrm{T}}A_i z)^{s_i}]$ for various s with $|s| = 10$.

s	$\mu = \mathbf{0}_m$	$\mu \neq \mathbf{0}_m$
(10)	42	481
(5,5)	554	15,310
(3,3,4)	3,895	143,795
(2,2,3,3)	16,561	673,364
(2,2,2,2,2)	60,775	2,572,407
(1,1,1,1,1,1,1,1,1,1)	1,436,714	63,673,506

where the sum extends over the positive divisors j of $\gcd(n_1, \dots, n_p)$ and ϕ is Euler's totient function. Karim, Sawada, Alamgir, and Husnine [11] give an efficient algorithm for *enumerating* bracelets with fixed content. Closed-form expressions for their number have a longer history: for two colors, the count solves the classical Reis problem and was obtained by Gupta [7], and the general case was settled recently by Hadjicostas [8], who counts bracelets with fixed content whose co-periods divide a given integer. The following lemma states the count in the form needed here, and a short, self-contained proof is given in Appendix A.

Lemma 8. *Let j denote the number of odd elements among $(n_1, \dots, n_p)$, and let $n = \sum_{i=1}^{p} n_i$. The number of bracelets with fixed content $(n_1, \dots, n_p)$ is*

$$B(n_1, \dots, n_p) = \begin{cases} \dfrac{1}{2} N_{\mathrm{neck}}(n_1, \dots, n_p) + \dfrac{((n-j)/2)!}{2\prod_{i=1}^{p}\lfloor n_i/2 \rfloor!}, & \text{if } j \le 2, \\ \dfrac{1}{2} N_{\mathrm{neck}}(n_1, \dots, n_p), & \text{if } j > 2, \end{cases} \tag{44}$$

and $n_\tau(\boldsymbol{\nu}) = B(n_1, \dots, n_p)$.

With Lemmas 7 and 8 in hand, we can count the distinct terms in the expansion of (37). Write $n(\boldsymbol{\nu}) = n_\theta(\boldsymbol{\nu}) + n_\tau(\boldsymbol{\nu})$ for the number of distinct terms that a block $\boldsymbol{\nu}$ can generate. For a multiset partition $\boldsymbol{\pi} \in \mathcal{B}_s$ with distinct blocks $\tilde{\boldsymbol{\pi}}_i$ of multiplicities h_i, the h_i copies of $\tilde{\boldsymbol{\pi}}_i$ jointly generate as many distinct products as there are multisets of size h_i drawn from $n(\tilde{\boldsymbol{\pi}}_i)$ objects, namely $\binom{n(\tilde{\boldsymbol{\pi}}_i)+h_i-1}{h_i}$. Hence the total number of distinct terms in the expansion of (37) is

$$M(s) = \sum_{\boldsymbol{\pi} \in \mathcal{B}_s} \prod_{i=1}^{r'} \binom{n(\tilde{\boldsymbol{\pi}}_i) + h_i - 1}{h_i}. \tag{45}$$

In the central case, the same formula applies with $n(\tilde{\boldsymbol{\pi}}_i)$ replaced by $n_\tau(\tilde{\boldsymbol{\pi}}_i)$. Table 7 reports $M(s)$ for various s with $|s| = 10$. The reduction relative to the case of distinct matrices is substantial; for instance, $\mathrm{E}[(z^{\mathrm{T}}A_1 z)^5 (z^{\mathrm{T}}A_2 z)^5]$ involves 15,310 distinct terms, against $M(10) = 63{,}673{,}506$ when all ten matrices are distinct.

8.3. *The explicit expression*

It remains to determine the multiplicity with which each distinct θ- or τ-term arises when the blocks of a set partition $\boldsymbol{\pi}'$ collapse to a block with repeated indices. For a word $\boldsymbol{\nu}$ with repeated entries, denote by $\boldsymbol{p}_j(\boldsymbol{\nu})$, $j = 1, \dots, n_\theta(\boldsymbol{\nu})$, the canonical θ-words formed from the entries of $\boldsymbol{\nu}$ (the multiset permutations that do not exceed their reversals lexicographically), and by $\boldsymbol{q}_j(\boldsymbol{\nu})$, $j = 1, \dots, n_\tau(\boldsymbol{\nu})$, the canonical τ-words (the bracelets with content $\boldsymbol{\nu}$). Recall that the *primitive period* of a word $\boldsymbol{w}$ of length n is the smallest positive integer ρ dividing n such that $\boldsymbol{w}$ is the concatenation of n/ρ copies of its first ρ entries. We say that a bracelet *maps to one necklace* if it coincides with some rotation of its reversal, and that it *maps to two necklaces* otherwise. For example, $(1, 2, 1, 2, 1, 2)$ has primitive period 2 and maps to one necklace (reversing gives $(2, 1, 2, 1, 2, 1)$, which is a rotation of the original), whereas $(1, 1, 2, 1, 2, 2)$ has primitive period 6 and maps to two necklaces.

Lemma 9. *Let $\boldsymbol{\nu}$ be a word of length $\ell = \ell(\boldsymbol{\nu})$ in which the integer l appears b_l times. Under the collapse of a block of ℓ distinct labels to $\boldsymbol{\nu}$, each canonical θ-word $\boldsymbol{p}_j(\boldsymbol{\nu})$ arises with multiplicity*

$$c_j = \begin{cases} \frac{1}{2} \prod_l b_l!, & \text{if } \boldsymbol{p}_j(\boldsymbol{\nu}) \text{ is palindromic and } \ell > 1, \\ \prod_l b_l!, & \text{otherwise,} \end{cases} \tag{46}$$

and each canonical τ-word $\boldsymbol{q}_j(\boldsymbol{\nu})$, with primitive period ρ_j, arises with multiplicity

$$d_j = \begin{cases} \frac{\rho_j}{2\ell} \prod_l b_l!, & \text{if } \boldsymbol{q}_j(\boldsymbol{\nu}) \text{ maps to one necklace and } \ell > 2, \\ \frac{\rho_j}{\ell} \prod_l b_l!, & \text{otherwise.} \end{cases} \tag{47}$$

Consequently, $\sum_j c_j = n_\theta(\ell)$ and $\sum_j d_j = n_\tau(\ell)$, with $n_\theta(\cdot)$ and $n_\tau(\cdot)$ as in Lemma 1: the multiplicities exactly exhaust the canonical words formed from ℓ distinct labels.

The block sum (22) generalizes accordingly: for a word $\boldsymbol{\nu}$, possibly with repeated entries, define

$$\lambda_{\boldsymbol{\nu}} = 2^{\mathbf{1}\{\ell(\boldsymbol{\nu})=2\}} \sum_{j=1}^{n_\theta(\boldsymbol{\nu})} c_j\, \theta_{\boldsymbol{p}_j(\boldsymbol{\nu})} + \sum_{j=1}^{n_\tau(\boldsymbol{\nu})} d_j\, \tau_{\boldsymbol{q}_j(\boldsymbol{\nu})}. \tag{48}$$

When the entries of $\boldsymbol{\nu}$ are distinct, all $c_j = d_j = 1$ and (48) reduces to (22).

Example 10. Let $\boldsymbol{\nu} = (1,1,2,2)$. From Example 9, the canonical θ-words are $(1,1,2,2)$, $(1,2,1,2)$, $(1,2,2,1)$, and $(2,1,1,2)$; the first two are not palindromic, so $c_1 = c_2 = 2!\,2! = 4$, and the last two are palindromic, so $c_3 = c_4 = 2$. The canonical τ-words are $(1,1,2,2)$, with primitive period 4, and $(1,2,1,2)$, with primitive period 2; both map to one necklace, so $d_1 = 4 \cdot 4/(2\cdot 4) = 2$ and $d_2 = 2\cdot 4/(2\cdot 4) = 1$. Hence

$$\lambda_{(1,1,2,2)} = 4\theta_{(1,1,2,2)} + 4\theta_{(1,2,1,2)} + 2\theta_{(1,2,2,1)} + 2\theta_{(2,1,1,2)} + 2\tau_{(1,1,2,2)} + \tau_{(1,2,1,2)}.$$

We can now state the main result of this section.

Proposition 4. *Let $\boldsymbol{z} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}$ positive semidefinite, let $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ be symmetric matrices, and let $s_1, \ldots, s_k$ be positive integers. Then*

$$\mathrm{E}\left[\prod_{i=1}^{k} (\boldsymbol{z}^{\mathrm{T}} \boldsymbol{A}_i \boldsymbol{z})^{s_i}\right] = \sum_{\boldsymbol{\pi}\in\mathcal{B}_{\boldsymbol{s}}} 2^{|\boldsymbol{s}|-f_1-f_2}\, c_{\boldsymbol{s}}(\boldsymbol{\pi}) \prod_{i=1}^{\ell(\boldsymbol{\pi})} \lambda_{\boldsymbol{\pi}_i}, \tag{49}$$

where f_1 and f_2 are the number of blocks of $\boldsymbol{\pi}$ with one and two elements, $c_{\boldsymbol{s}}(\boldsymbol{\pi})$ is given by Lemma 6, and $\lambda_{\boldsymbol{\pi}_i}$ is given by (48). If $\boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and the $\boldsymbol{A}_i$ are Hermitian, then (49) holds for $\mathrm{E}[\prod_{i=1}^{k} (\boldsymbol{z}^{\mathrm{H}} \boldsymbol{A}_i \boldsymbol{z})^{s_i}]$ with $2^{|\boldsymbol{s}|-f_1-f_2}$ replaced by $2^{\ell(\boldsymbol{\pi})-f_1-f_2}$ and with the real parts taken in $\theta_{\boldsymbol{\nu}}$ and $\tau_{\boldsymbol{\nu}}$, as in (27) and (28).

Remark 1. The factors $\prod_j b_{ij}!$ appearing in the denominator of $c_{\boldsymbol{s}}(\boldsymbol{\pi})$ in (39) cancel against the factors $\prod_l b_l!$ appearing in the multiplicities (46) and (47), one block at a time. In an actual implementation, both sets of factors can therefore be dropped.

Example 11. Let $s_1 = s_2 = 2$. Table 8 provides all the quantities required in (49), using the multiset partitions listed in Table 6, the multiplicities $c_{\boldsymbol{s}}(\boldsymbol{\pi})$ from Example 8, and the block sums from Lemma 9 and Example 10. Reading off the table,

$$\begin{aligned} \mathrm{E}\left[(\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{z})^2(\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}_2\boldsymbol{z})^2\right] &= (\theta_1+\tau_1)^2(\theta_2+\tau_2)^2 + 2(\theta_1+\tau_1)^2\left[2\theta_{(2,2)}+\tau_{(2,2)}\right] \\ &+ 8(\theta_1+\tau_1)(\theta_2+\tau_2)\left[2\theta_{(1,2)}+\tau_{(1,2)}\right] + 2(\theta_2+\tau_2)^2\left[2\theta_{(1,1)}+\tau_{(1,1)}\right] \\ &+ 4\left[2\theta_{(1,1)}+\tau_{(1,1)}\right]\left[2\theta_{(2,2)}+\tau_{(2,2)}\right] + 8\left[2\theta_{(1,2)}+\tau_{(1,2)}\right]^2 \\ &+ 16(\theta_1+\tau_1)\left[2\theta_{(1,2,2)}+\theta_{(2,1,2)}+\tau_{(1,2,2)}\right] \\ &+ 16(\theta_2+\tau_2)\left[2\theta_{(1,1,2)}+\theta_{(1,2,1)}+\tau_{(1,1,2)}\right] \\ &+ 16\left[4\theta_{(1,1,2,2)}+4\theta_{(1,2,1,2)}+2\theta_{(1,2,2,1)}+2\theta_{(2,1,1,2)}+2\tau_{(1,1,2,2)}+\tau_{(1,2,1,2)}\right]. \end{aligned}$$

Table 8: Construction of the explicit expression of $\mathrm{E}[(z^{\mathrm{T}}A_1z)^2(z^{\mathrm{T}}A_2z)^2]$ for $z \sim \mathcal{N}(\mu, \Sigma)$ using (49).

$\pi \in \mathcal{B}_{(2,2)}$	$2^{\lvert s\rvert - f_1 - f_2}$	$c_s(\pi)$	λ_{π_1}	λ_{π_2}	λ_{π_3}	λ_{π_4}
(1)(1)(2)(2)	1	1	$\theta_1 + \tau_1$	$\theta_1 + \tau_1$	$\theta_2 + \tau_2$	$\theta_2 + \tau_2$
(1)(1)(2,2)	2	1	$\theta_1 + \tau_1$	$\theta_1 + \tau_1$	$2\theta_{(2,2)} + \tau_{(2,2)}$	–
(1)(2)(1,2)	2	4	$\theta_1 + \tau_1$	$\theta_2 + \tau_2$	$2\theta_{(1,2)} + \tau_{(1,2)}$	–
(2)(2)(1,1)	2	1	$\theta_2 + \tau_2$	$\theta_2 + \tau_2$	$2\theta_{(1,1)} + \tau_{(1,1)}$	–
(1,1)(2,2)	4	1	$2\theta_{(1,1)} + \tau_{(1,1)}$	$2\theta_{(2,2)} + \tau_{(2,2)}$	–	–
(1,2)(1,2)	4	2	$2\theta_{(1,2)} + \tau_{(1,2)}$	$2\theta_{(1,2)} + \tau_{(1,2)}$	–	–
(1)(1,2,2)	8	2	$\theta_1 + \tau_1$	$2\theta_{(1,2,2)} + \theta_{(2,1,2)} + \tau_{(1,2,2)}$	–	–
(2)(1,1,2)	8	2	$\theta_2 + \tau_2$	$2\theta_{(1,1,2)} + \theta_{(1,2,1)} + \tau_{(1,1,2)}$	–	–
(1,1,2,2)	16	1	$4\theta_{(1,1,2,2)} + 4\theta_{(1,2,1,2)} + 2\theta_{(1,2,2,1)} + 2\theta_{(2,1,1,2)} + 2\tau_{(1,1,2,2)} + \tau_{(1,2,1,2)}$			

Table 9: Construction of the explicit expression of $\mathrm{E}[\mathrm{tr}(WA_1)\,\mathrm{tr}(WA_2)^2]$ for $W \sim \mathcal{CW}_m(n, \Sigma, \Lambda)$.

$\pi \in \mathcal{B}_{(1,2)}$	$2^{\ell(\pi) - f_1 - f_2}$	$c_s(\pi)$	λ_{π_1}	λ_{π_2}	λ_{π_3}
(1)(2)(2)	1	1	$\theta_1 + \tau_1$	$\theta_2 + \tau_2$	$\theta_2 + \tau_2$
(1)(2,2)	1	1	$\theta_1 + \tau_1$	$2\theta_{(2,2)} + \tau_{(2,2)}$	–
(2)(1,2)	1	2	$\theta_2 + \tau_2$	$2\theta_{(1,2)} + \tau_{(1,2)}$	–
(1,2,2)	2	1	$2\theta_{(1,2,2)} + \theta_{(2,1,2)} + \tau_{(1,2,2)}$	–	–

Expanding the right-hand side yields 54 distinct terms, in agreement with $M((2,2)) = 54$ and significantly fewer than the $M(4) = 123$ terms required when all four matrices are distinct.

Combining Proposition 4 with the Wishart reduction of Section 7 yields the product moments $\mathrm{E}[\prod_{i=1}^{k} \mathrm{tr}(WA_i)^{s_i}]$ for real and complex, central and noncentral Wishart matrices. We close with the example that corrects the expression of Di Nardo [2].

Example 12. Let $W \sim \mathcal{CW}_m(n, \Sigma, \Lambda)$ and consider $\mathrm{E}[\mathrm{tr}(WA_1)\,\mathrm{tr}(WA_2)^2]$, i.e., $s = (1,2)$. Table 9 provides the required quantities, with the θ- and τ-terms given, per Corollary 3, by

$$\theta_\nu = \Re\left(\mathrm{tr}(A_{\nu_1}\Sigma \cdots \Sigma A_{\nu_p}\Lambda)\right), \qquad \tau_\nu = \Re\left(n\,\mathrm{tr}(A_{\nu_1}\Sigma \cdots \Sigma A_{\nu_p}\Sigma)\right).$$

Reading off the table,

$$\begin{aligned}\mathrm{E}\left[\mathrm{tr}(WA_1)\,\mathrm{tr}(WA_2)^2\right] &= (\theta_1 + \tau_1)(\theta_2 + \tau_2)^2 + (\theta_1 + \tau_1)\left[2\theta_{(2,2)} + \tau_{(2,2)}\right] \\ &\quad + 2(\theta_2 + \tau_2)\left[2\theta_{(1,2)} + \tau_{(1,2)}\right] + 2\left[2\theta_{(1,2,2)} + \theta_{(2,1,2)} + \tau_{(1,2,2)}\right].\end{aligned}$$

9. Concluding remarks

We have derived an explicit expression for the expectation of a product of quadratic forms in normal random vectors with arbitrary mean and positive semidefinite covariance matrix, in both the real and the complex case. The expression contains no repeated terms—so it has the minimum possible number of terms—and every coefficient is an explicit power of two. A companion expression indexed by set partitions makes the enumeration computationally practical, and further extensions deliver the product moments of central and noncentral, real and complex Wishart distributions, as well as products in which quadratic forms are repeated. As a by-product, we also obtained exact counts of the number of terms in each of our expansions. MATLAB implementations of all the formulas in this paper—including routines that generate the symbolic expansions themselves—are available at `https://www-2.rotman.utoronto.ca/~kan/research.htm`.

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Appendix A. Proofs

Throughout the proofs, "term" refers to a formal product of symbols $\theta_{\boldsymbol{\nu}}$ and $\tau_{\boldsymbol{\nu}}$ indexed by canonical words, and two terms are "distinct" if they differ as formal products; for particular numerical choices of $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_k)$, distinct terms may of course take equal values.

Appendix A.1. Proof of Lemma 1

For $i = 1$ both counts are trivially 1, and for $i = 2$ there is a single τ-term because $\tau_{(a,b)} = \tau_{(b,a)}$ by (4). Consider $i \geq 2$ arrangements for θ. The $i!$ arrangements of the i distinct entries split into pairs $\{\boldsymbol{\nu}, \boldsymbol{\nu}^R\}$; no arrangement is its own reversal, because the first and last entries are distinct. Since $\theta_{\boldsymbol{\nu}} = \theta_{\boldsymbol{\nu}^R}$ and arrangements in different pairs produce different index words, the number of distinct θ-terms is $i!/2$.

For τ with $i \geq 3$, the dihedral group of order $2i$ acts on arrangements by rotations and reversal, and $\tau_{\boldsymbol{\nu}}$ is constant on orbits by (4). The action is free: a nontrivial rotation fixes no arrangement of distinct entries, and a reversal composed with a rotation would identify the cyclic sequence with its mirror image, which is impossible for $i \geq 3$ distinct entries because the two neighbors of any fixed entry appear in swapped order. Hence every orbit has exactly $2i$ elements, and the number of orbits, i.e., of distinct τ-terms, is $i!/(2i) = (i-1)!/2$. □

Appendix A.2. Proof of Lemma 2

Write $\boldsymbol{\Sigma} = \boldsymbol{C}\boldsymbol{C}^{\mathrm{T}}$ and $\boldsymbol{z} = \boldsymbol{\mu} + \boldsymbol{C}\boldsymbol{x}$ with $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}_m, \mathbf{I}_m)$, and let $\boldsymbol{G} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{C}$, so that $\mathrm{tr}(\boldsymbol{G}^j) = \mathrm{tr}((\boldsymbol{A}\boldsymbol{\Sigma})^j) = \tilde{\tau}_j$. For a symmetric matrix $\boldsymbol{M}$ with $\mathbf{I}_m - 2\boldsymbol{M}$ positive definite and a vector $\boldsymbol{b}$, the Gaussian integral

$$\mathrm{E}\left[\exp\left(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{x}\right)\right] = |\mathbf{I}_m - 2\boldsymbol{M}|^{-1/2} \exp\left(\tfrac{1}{2}\boldsymbol{b}^{\mathrm{T}}(\mathbf{I}_m - 2\boldsymbol{M})^{-1}\boldsymbol{b}\right)$$

applied with $\boldsymbol{M} = t\boldsymbol{G}$ and $\boldsymbol{b} = 2t\boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{\mu}$ gives, for all t in a neighborhood of zero,

$$K(t) := \log \mathrm{E}\left[e^{t\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{z}}\right] = -\frac{1}{2}\log|\mathbf{I}_m - 2t\boldsymbol{G}| + t\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{\mu} + 2t^2\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{C}(\mathbf{I}_m - 2t\boldsymbol{G})^{-1}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{\mu}. \tag{A.1}$$

Using $-\log|\mathbf{I}_m - 2t\boldsymbol{G}| = \sum_{j\geq 1}(2t)^j\,\mathrm{tr}(\boldsymbol{G}^j)/j$ and

$$\boldsymbol{A}\boldsymbol{C}\boldsymbol{G}^l\boldsymbol{C}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{\Sigma}\boldsymbol{A})^l\boldsymbol{\Sigma}\boldsymbol{A} \qquad \Longrightarrow \qquad \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{C}\boldsymbol{G}^l\boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{\mu} = \tilde{\theta}_{l+2},$$

we obtain the convergent expansion

$$K(t) = \sum_{j\geq 1}\frac{2^{j-1}\tilde{\tau}_j}{j}t^j + \tilde{\theta}_1 t + \sum_{j\geq 2}2^{j-1}\tilde{\theta}_j t^j = \sum_{j\geq 1}\left[2^{j-1}(j-1)!\left(\tilde{\tau}_j + j\tilde{\theta}_j\right)\right]\frac{t^j}{j!},$$

and (8) follows by reading off the j-th cumulant $\psi_j = K^{(j)}(0)$. The argument requires only $\boldsymbol{\Sigma} = \boldsymbol{C}\boldsymbol{C}^{\mathrm{T}}$, so it covers positive semidefinite $\boldsymbol{\Sigma}$. □

Appendix A.3. Proof of Lemma 3

Let $X = \boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{z}$, with moment generating function $M(t) = e^{K(t)}$, which is finite in a neighborhood of zero. Writing $K(t) = \sum_{j\geq 1}\psi_j t^j/j!$ and expanding the exponential,

$$M(t) = \sum_{r\geq 0}\frac{1}{r!}\left(\sum_{j\geq 1}\psi_j\frac{t^j}{j!}\right)^r = \sum_{r\geq 0}\frac{1}{r!}\sum_{j_1,\ldots,j_r\geq 1}\frac{\psi_{j_1}\cdots\psi_{j_r}}{j_1!\cdots j_r!}t^{j_1+\cdots+j_r}.$$

Extracting the coefficient of t^k and multiplying by $k!$,

$$\mathrm{E}[X^k] = \sum_{r\geq 1} \frac{1}{r!} \sum_{\substack{j_1,\ldots,j_r\geq 1 \\ j_1+\cdots+j_r=k}} \binom{k}{j_1,\ldots,j_r} \psi_{j_1}\cdots\psi_{j_r} = \sum_{\pi\in\mathcal{B}_k} \prod_{b\in\pi} \psi_{|b|},$$

where the last equality holds because the number of ways to partition $\{1,\ldots,k\}$ into an *ordered* list of r blocks of sizes $j_1,\ldots,j_r$ is the multinomial coefficient $\binom{k}{j_1,\ldots,j_r}$, and each unordered set partition with r blocks corresponds to exactly $r!$ ordered lists (the blocks are pairwise distinct as sets). Grouping the set partitions of $\{1,\ldots,k\}$ by their type $\boldsymbol{\kappa} \vdash k$, of which there are $q(\boldsymbol{\kappa}) = k!/\prod_{i=1}^k (i!)^{f_i} f_i!$ each, and substituting (8),

$$\mathrm{E}[X^k] = \sum_{\boldsymbol{\kappa}\vdash k} q(\boldsymbol{\kappa}) \prod_{i=1}^k \psi_i^{f_i} = \sum_{\boldsymbol{\kappa}\vdash k} q(\boldsymbol{\kappa}) \prod_{i=1}^k \left[2^{i-1}(i-1)!\right]^{f_i} \left(\tilde{\tau}_i + i\tilde{\theta}_i\right)^{f_i} = \sum_{\boldsymbol{\kappa}\vdash k} \frac{k!\,2^k}{\prod_{i=1}^k f_i!\,(2i)^{f_i}} \prod_{i=1}^k \left(i\tilde{\theta}_i + \tilde{\tau}_i\right)^{f_i},$$

where the last step uses $q(\boldsymbol{\kappa})\prod_i [2^{i-1}(i-1)!]^{f_i} = k!\prod_i 2^{(i-1)f_i}/(i^{f_i} f_i!) = k!\,2^k/\prod_i f_i!(2i)^{f_i}$, since $\sum_i (i-1)f_i = k - \sum_i f_i$. This proves (9), and (10) follows by applying the binomial expansion

$$\left(i\tilde{\theta}_i + \tilde{\tau}_i\right)^{f_i} = \sum_{g_i=0}^{f_i} \binom{f_i}{g_i} i^{g_i}\, \tilde{\theta}_i^{g_i} \tilde{\tau}_i^{f_i-g_i}$$

to each factor. Finally, the monomial $\prod_i \tilde{\theta}_i^{g_i}\tilde{\tau}_i^{f_i-g_i}$ determines $(\boldsymbol{f},\boldsymbol{g})$ uniquely (its exponents recover g_i and $f_i - g_i$), so no two terms of (10) coincide, and for each $\boldsymbol{\kappa}$ the number of admissible $\boldsymbol{g}$ vectors is $\prod_i (1+f_i)$, which proves (12). □

Appendix A.4. Proof of Lemma 4

By multilinearity,

$$P\left(\sum_{j=1}^k t_j \boldsymbol{A}_j\right) = \Phi\left(\sum_{j_1=1}^k t_{j_1}\boldsymbol{A}_{j_1},\ldots,\sum_{j_k=1}^k t_{j_k}\boldsymbol{A}_{j_k}\right) = \sum_{j_1=1}^k \cdots \sum_{j_k=1}^k t_{j_1}\cdots t_{j_k}\,\Phi(\boldsymbol{A}_{j_1},\ldots,\boldsymbol{A}_{j_k}).$$

The monomial $t_1t_2\cdots t_k$ arises exactly from the index vectors $(j_1,\ldots,j_k)$ that are permutations of $(1,\ldots,k)$, and by the symmetry of Φ each such vector contributes $\Phi(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_k)$. As there are $k!$ permutations, (14) follows. For the final claim, if two symmetric multilinear functions have the same diagonal restriction P, then (14) shows that they agree everywhere. □

Appendix A.5. Proof of Proposition 1

Let $\alpha(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_k) = \mathrm{E}[\prod_{i=1}^k \boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}_i\boldsymbol{z}]$. Since each $\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{z}$ is linear in $\boldsymbol{A}$ and expectation is linear, α is a symmetric multilinear function of $(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_k)$, with diagonal restriction $P(\boldsymbol{A}) = \mathrm{E}[(\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{z})^k]$. The proof proceeds in three steps.

Step 1: polarization. Let $\boldsymbol{A}(\boldsymbol{t}) = t_1\boldsymbol{A}_1 + \cdots + t_k\boldsymbol{A}_k$. By Lemma 4,

$$k!\,\alpha(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_k) = [t_1\cdots t_k]\,\mathrm{E}\left[(\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}(\boldsymbol{t})\boldsymbol{z})^k\right]. \tag{A.2}$$

By Lemma 3, the right-hand side of (A.2) can be computed from (10) applied to $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{t})$. The quantities (7) become polynomials in $\boldsymbol{t}$:

$$\tilde{\tau}_l(\boldsymbol{A}(\boldsymbol{t})) = \mathrm{tr}\left[\left(\textstyle\sum_{j=1}^k t_j\boldsymbol{A}_j\boldsymbol{\Sigma}\right)^l\right] = \sum_{\boldsymbol{\nu}\in\{1,\ldots,k\}^l} t_{\nu_1}\cdots t_{\nu_l}\,\tau_{\boldsymbol{\nu}}, \tag{A.3}$$

$$\tilde{\theta}_l(\boldsymbol{A}(\boldsymbol{t})) = \boldsymbol{\mu}^{\mathrm{T}}\left(\textstyle\sum_j t_j\boldsymbol{A}_j\boldsymbol{\Sigma}\right)^{l-1}\left(\textstyle\sum_j t_j\boldsymbol{A}_j\right)\boldsymbol{\mu} = \sum_{\boldsymbol{\nu}\in\{1,\ldots,k\}^l} t_{\nu_1}\cdots t_{\nu_l}\,\theta_{\boldsymbol{\nu}}, \tag{A.4}$$

i.e., $\tilde{\tau}_l(\boldsymbol{A}(\boldsymbol{t}))$ and $\tilde{\theta}_l(\boldsymbol{A}(\boldsymbol{t}))$ are generating functions of the τ- and θ-terms whose index words have length l.

Step 2: coefficient extraction. Fix $\boldsymbol{\kappa} \vdash k$ (equivalently $\boldsymbol{f}$) and $\mathbf{0} \le \boldsymbol{g} \le \boldsymbol{f}$. The corresponding term of (10) is $c_{\boldsymbol{f},\boldsymbol{g}} \prod_{l=1}^{k} \tilde{\theta}_l(\boldsymbol{A}(\boldsymbol{t}))^{g_l} \tilde{\tau}_l(\boldsymbol{A}(\boldsymbol{t}))^{f_l-g_l}$, a homogeneous polynomial in $\boldsymbol{t}$ of degree $\sum_l l f_l = k$. Expanding the product of the $r = \ell(\boldsymbol{\kappa})$ generating-function factors—ordered so that the factors with $l = 1$ come first, then $l = 2$, and so on, with the θ-factors preceding the τ-factors within each group—a monomial is obtained by selecting one word $\boldsymbol{\nu}^{(1)}, \dots, \boldsymbol{\nu}^{(r)}$ from each factor. The selection contributes to the coefficient of $t_1 \cdots t_k$ if and only if the concatenation $(\boldsymbol{\nu}^{(1)}, \dots, \boldsymbol{\nu}^{(r)})$ uses each index in $\{1, \dots, k\}$ exactly once, i.e., if and only if it is a permutation $\boldsymbol{\pi} \in \mathcal{S}_k$ split into blocks in exactly the manner prescribed above (15); the value contributed is then $\eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}}$. Summing over $(\boldsymbol{f}, \boldsymbol{g})$ and using (A.2),

$$\alpha(\boldsymbol{A}_1, \dots, \boldsymbol{A}_k) = \sum_{\boldsymbol{\kappa} \vdash k} \sum_{\mathbf{0} \le \boldsymbol{g} \le \boldsymbol{f}} \frac{c_{\boldsymbol{f},\boldsymbol{g}}}{k!} \sum_{\boldsymbol{\pi} \in \mathcal{S}_k} \eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}}, \tag{A.5}$$

where $\mathcal{S}_k$ denotes the full symmetric group on $\{1, \dots, k\}$.

Step 3: reduction to admissible permutations. Expression (A.5) contains repeated terms. Fix $(\boldsymbol{f}, \boldsymbol{g})$ and declare $\boldsymbol{\pi} \approx \boldsymbol{\pi}'$ if $\boldsymbol{\pi}'$ can be obtained from $\boldsymbol{\pi}$ by a finite sequence of the following moves: (i) rotating or reversing a τ-designated block; (ii) reversing a θ-designated block; (iii) exchanging two entire blocks of equal length and equal designation. By (4) and the commutativity of scalar multiplication, $\eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}} = \eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}'}$ whenever $\boldsymbol{\pi} \approx \boldsymbol{\pi}'$.

We claim each equivalence class has exactly

$$\omega_{\boldsymbol{f},\boldsymbol{g}} = \prod_{i=1}^{k} \left[g_i!\, \delta_i^{g_i}\right] \left[(f_i - g_i)!\, \sigma_i^{f_i - g_i}\right], \qquad \delta_i = \begin{cases} 1, & i = 1, \\ 2, & i \ge 2, \end{cases} \qquad \sigma_i = \begin{cases} i, & i \le 2, \\ 2i, & i \ge 3, \end{cases} \tag{A.6}$$

elements. Indeed, the moves act independently on the different blocks and block positions: a θ-designated block of length i has δ_i distinct arrangements in its move-orbit (itself and its reversal, which differ when $i \ge 2$ because the entries are distinct); a τ-designated block of length i has σ_i distinct arrangements (its rotations and reflected rotations, which, as in the proof of Lemma 1, are pairwise distinct when $i \ge 3$, coincide pairwise when $i = 2$, and are trivial when $i = 1$); and the g_i (respectively $f_i - g_i$) same-designation blocks of length i can be permuted among their positions in $g_i!$ (respectively $(f_i - g_i)!$) ways, all yielding distinct permutations because the blocks have pairwise distinct contents.

Each class contains exactly one admissible permutation: within a class, each block can be put in canonical form in exactly one way (each rotation–reversal class of words with distinct entries contains exactly one canonical word), and the same-length, same-designation blocks can be sorted by their first elements in exactly one way. Hence

$$\sum_{\boldsymbol{\pi} \in \mathcal{S}_k} \eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}} = \omega_{\boldsymbol{f},\boldsymbol{g}} \sum_{\boldsymbol{\pi} \in \mathcal{S}_{\boldsymbol{f},\boldsymbol{g}}} \eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}},$$

and it remains to evaluate the coefficient $c_{\boldsymbol{f},\boldsymbol{g}}\, \omega_{\boldsymbol{f},\boldsymbol{g}}/k!$. From (11) and (A.6), and using $\binom{f_i}{g_i} g_i!(f_i - g_i)! = f_i!$,

$$\frac{c_{\boldsymbol{f},\boldsymbol{g}}\, \omega_{\boldsymbol{f},\boldsymbol{g}}}{k!} = 2^k \prod_{i=1}^{k} \frac{i^{g_i} \delta_i^{g_i} \sigma_i^{f_i - g_i}}{(2i)^{f_i}} = 2^k \prod_{i=1}^{k} \left(\frac{i\, \delta_i}{2i}\right)^{g_i} \left(\frac{\sigma_i}{2i}\right)^{f_i - g_i} = 2^k \cdot 2^{-g_1} \cdot 2^{-(f_1 - g_1)} \cdot 2^{-(f_2 - g_2)} = 2^{k - f_1 - f_2 + g_2},$$

because $i\delta_i/(2i) = \delta_i/2$ equals $1/2$ for $i = 1$ and 1 for $i \ge 2$, while $\sigma_i/(2i)$ equals $1/2$ for $i \le 2$ and 1 for $i \ge 3$. This establishes (16).

For part (i), note that a term $\eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}}$ with $\boldsymbol{\pi} \in \mathcal{S}_{\boldsymbol{f},\boldsymbol{g}}$ is a formal product of symbols $\theta_{\boldsymbol{w}}$ and $\tau_{\boldsymbol{w}}$ indexed by canonical words $\boldsymbol{w}$ whose contents partition $\{1, \dots, k\}$. From the term one recovers the multiset of pairs (designation, word), hence the block sizes $\boldsymbol{f}$, the designation counts $\boldsymbol{g}$, and, after sorting, the admissible permutation $\boldsymbol{\pi}$ itself. Distinct triples therefore yield distinct terms.

For part (ii), an admissible permutation is constructed by (a) choosing an unordered set partition of $\{1, \dots, k\}$ of type $\boldsymbol{\kappa}$, in one of $q(\boldsymbol{\kappa})$ ways, where $q(\boldsymbol{\kappa})$ counts the ways of placing k labeled elements into f_i unordered bins of size i for each i; (b) choosing, for each i, which g_i of the f_i blocks of length i are θ-designated, in $\prod_i \binom{f_i}{g_i}$ ways; and (c) choosing a canonical word for each block, in $n_\theta(i)^{g_i} n_\tau(i)^{f_i - g_i}$ ways for the blocks of length i (Lemma 1); the ordering of the blocks is then forced. This gives (17), and summing over $\boldsymbol{g}$ with the binomial theorem, $\sum_{\mathbf{0} \le \boldsymbol{g} \le \boldsymbol{f}} h_{\boldsymbol{f},\boldsymbol{g}} = q(\boldsymbol{\kappa}) \prod_i [n_\theta(i) + n_\tau(i)]^{f_i}$, which proves (18). □

Appendix A.6. Proof of Corollary 1

When $\boldsymbol{\mu} = \mathbf{0}_m$, every θ-term vanishes, so the terms of (16) with $\boldsymbol{g} \neq \mathbf{0}$ drop out, and for $\boldsymbol{g} = \mathbf{0}$ we have $g_2 = 0$ and $\eta_{\boldsymbol{f},\mathbf{0},\boldsymbol{\pi}} = \tau_{\pi_1} \cdots \tau_{\pi_r}$ with $\mathcal{S}_{\boldsymbol{f},\mathbf{0}} = \mathcal{S}_{\boldsymbol{\kappa}}$, which yields (20). For the term count, (17) with $\boldsymbol{g} = \mathbf{0}$ and Lemma 1 give

$$\varphi(\boldsymbol{\kappa}) = h_{\boldsymbol{f},\mathbf{0}} = q(\boldsymbol{\kappa}) \prod_{i=1}^{k} n_\tau(i)^{f_i} = \frac{k!}{\prod_{i=1}^{k} (i!)^{f_i} f_i!} \prod_{i=1}^{k} n_\tau(i)^{f_i} = \frac{k!\, 2^{f_1+f_2}}{\prod_{i=1}^{k} f_i!\, (2i)^{f_i}},$$

where the last equality uses $n_\tau(i)/i! = 2^{\mathbf{1}\{i\leq 2\}}/(2i)$ for every $i \geq 1$. □

Appendix A.7. Proof of Proposition 2

Consider the map that sends a triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{\pi})$ with $\boldsymbol{\pi} \in \mathcal{S}_{\boldsymbol{f},\boldsymbol{g}}$ to the pair consisting of (a) the set partition $\sigma(\boldsymbol{\pi}) \in \mathcal{B}_k$ whose blocks are the contents of the blocks of $\boldsymbol{\pi}$, and (b) the assignment, to each block of $\sigma(\boldsymbol{\pi})$, of its designation (θ or τ) and its canonical word. By construction of $\mathcal{S}_{\boldsymbol{f},\boldsymbol{g}}$ (steps (a)–(c) in the proof of Proposition 1(ii)), this map is a bijection onto the set of pairs (set partition of $\{1, \ldots, k\}$, per-block choice of a designation and a canonical word), with inverse obtained by reading off $\boldsymbol{f}$ from the block sizes, $\boldsymbol{g}$ from the designation counts, and sorting blocks canonically. Under this bijection, $\eta_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}}$ equals the product over the blocks of the selected θ- or τ-symbol.

Now group the right-hand side of (16) by $\sigma(\boldsymbol{\pi}) = \boldsymbol{\pi}^* \in \mathcal{B}_k$. For fixed $\boldsymbol{\pi}^*$ with block sizes giving frequencies f_1, f_2 (for sizes one and two), the coefficient of a term whose θ-designated blocks of size two number g_2 is $2^{k-f_1-f_2+g_2} = 2^{k-f_1-f_2} \cdot 2^{g_2}$. Distributing the factor 2^{g_2} as a factor of 2 to each θ-designated block of size exactly two, the sum over all designation-and-word choices for the blocks factorizes as the product of per-block sums, which are precisely the block sums $\lambda_{\pi_i^*}$ of (22). This proves (23), and the bijection shows that expanding (23) reproduces the terms of (16) with identical coefficients. The central case follows by dropping the θ-terms from (22). □

Appendix A.8. Proof of Lemma 5

For complex matrices $\boldsymbol{A} = \boldsymbol{A}_r + \mathrm{i}\boldsymbol{A}_c$ and $\boldsymbol{B} = \boldsymbol{B}_r + \mathrm{i}\boldsymbol{B}_c$ with product $\boldsymbol{C} = \boldsymbol{AB} = \boldsymbol{C}_r + \mathrm{i}\boldsymbol{C}_c$, where $\boldsymbol{C}_r = \boldsymbol{A}_r\boldsymbol{B}_r - \boldsymbol{A}_c\boldsymbol{B}_c$ and $\boldsymbol{C}_c = \boldsymbol{A}_r\boldsymbol{B}_c + \boldsymbol{A}_c\boldsymbol{B}_r$, a direct computation shows that the augmented representation (25) is multiplicative:

$$\hat{\boldsymbol{A}}\hat{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{A}_r & -\boldsymbol{A}_c \\ \boldsymbol{A}_c & \boldsymbol{A}_r \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_r & -\boldsymbol{B}_c \\ \boldsymbol{B}_c & \boldsymbol{B}_r \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}_r & -\boldsymbol{C}_c \\ \boldsymbol{C}_c & \boldsymbol{C}_r \end{bmatrix} = \hat{\boldsymbol{C}}.$$

Since $\hat{\boldsymbol{\Sigma}}$ in (24) is one half of the augmented representation of $\boldsymbol{\Sigma}$, induction on the number of factors gives

$$\hat{\boldsymbol{A}}_{\nu_1}\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{A}}_{\nu_2}\hat{\boldsymbol{\Sigma}} \cdots \hat{\boldsymbol{A}}_{\nu_p}\hat{\boldsymbol{\Sigma}} = \frac{1}{2^p} \begin{bmatrix} \boldsymbol{C}_r & -\boldsymbol{C}_c \\ \boldsymbol{C}_c & \boldsymbol{C}_r \end{bmatrix}, \qquad \boldsymbol{C} = \boldsymbol{A}_{\nu_1}\boldsymbol{\Sigma}\boldsymbol{A}_{\nu_2}\boldsymbol{\Sigma} \cdots \boldsymbol{A}_{\nu_p}\boldsymbol{\Sigma} = \boldsymbol{C}_r + \mathrm{i}\boldsymbol{C}_c.$$

Taking traces, $\hat{\tau}_{\boldsymbol{\nu}} = 2\,\mathrm{tr}(\boldsymbol{C}_r)/2^p = \Re(\mathrm{tr}\,\boldsymbol{C})/2^{p-1} = \tau_{\boldsymbol{\nu}}/2^{p-1}$. Similarly, with $\boldsymbol{D} = \boldsymbol{A}_{\nu_1}\boldsymbol{\Sigma} \cdots \boldsymbol{\Sigma}\boldsymbol{A}_{\nu_p} = \boldsymbol{D}_r + \mathrm{i}\boldsymbol{D}_c$ (a product with $p-1$ factors of $\boldsymbol{\Sigma}$),

$$\hat{\theta}_{\boldsymbol{\nu}} = \frac{1}{2^{p-1}} \begin{bmatrix} \boldsymbol{\mu}_r \\ \boldsymbol{\mu}_c \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{D}_r & -\boldsymbol{D}_c \\ \boldsymbol{D}_c & \boldsymbol{D}_r \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}_r \\ \boldsymbol{\mu}_c \end{bmatrix} = \frac{\boldsymbol{\mu}_r^{\mathrm{T}}\boldsymbol{D}_r\boldsymbol{\mu}_r + \boldsymbol{\mu}_c^{\mathrm{T}}\boldsymbol{D}_r\boldsymbol{\mu}_c - \boldsymbol{\mu}_r^{\mathrm{T}}\boldsymbol{D}_c\boldsymbol{\mu}_c + \boldsymbol{\mu}_c^{\mathrm{T}}\boldsymbol{D}_c\boldsymbol{\mu}_r}{2^{p-1}} = \frac{\Re\,(\boldsymbol{\mu}^{\mathrm{H}}\boldsymbol{D}\boldsymbol{\mu})}{2^{p-1}} = \frac{\theta_{\boldsymbol{\nu}}}{2^{p-1}},$$

where the penultimate equality follows by expanding $\boldsymbol{\mu}^{\mathrm{H}}\boldsymbol{D}\boldsymbol{\mu} = (\boldsymbol{\mu}_r - \mathrm{i}\boldsymbol{\mu}_c)^{\mathrm{T}}(\boldsymbol{D}_r + \mathrm{i}\boldsymbol{D}_c)(\boldsymbol{\mu}_r + \mathrm{i}\boldsymbol{\mu}_c)$ and collecting the real part. □

Appendix A.9. Proof of Proposition 3

By (26), $\mathrm{E}[\prod_{i=1}^k \boldsymbol{z}^{\mathrm{H}}\boldsymbol{A}_i\boldsymbol{z}]$ equals the expectation of a product of quadratic forms in $\hat{\boldsymbol{z}} \sim \mathcal{N}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}})$ with symmetric matrices $\hat{\boldsymbol{A}}_1, \ldots, \hat{\boldsymbol{A}}_k$, to which Proposition 1 applies:

$$\mathrm{E}\left[\prod_{i=1}^{k} \boldsymbol{z}^{\mathrm{H}}\boldsymbol{A}_i\boldsymbol{z}\right] = \sum_{\boldsymbol{\kappa}\vdash k} \sum_{\mathbf{0}\leq\boldsymbol{g}\leq\boldsymbol{f}} 2^{k-f_1-f_2+g_2} \sum_{\boldsymbol{\pi}\in\mathcal{S}_{\boldsymbol{f},\boldsymbol{g}}} \hat{\eta}_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{\pi}},$$

where $\hat{\eta}$ is built from the augmented terms $\hat{\theta}_{\nu}$ and $\hat{\tau}_{\nu}$. Each $\hat{\eta}_{f,g,\pi}$ is a product of $r = \ell(\boldsymbol{\kappa})$ factors, one for each block; by Lemma 5, a block of length i contributes a factor $2^{-(i-1)}$ relative to the corresponding complex term, so

$$\hat{\eta}_{f,g,\pi} = 2^{-\sum_i (i-1) f_i}\, \eta_{f,g,\pi} = 2^{-(k-r)}\, \eta_{f,g,\pi},$$

with η now built from (27) and (28). Absorbing $2^{-(k-r)}$ into the coefficient yields $2^{k-f_1-f_2+g_2-(k-r)} = 2^{\ell(\boldsymbol{\kappa})-f_1-f_2+g_2}$, which is (30). The set-partition form (31) follows by the identical regrouping as in the proof of Proposition 2, applied to (30). □

Appendix A.10. Proof of Corollary 2

Let $\boldsymbol{y} = (\boldsymbol{z}_1^{\mathrm{T}}, \ldots, \boldsymbol{z}_n^{\mathrm{T}})^{\mathrm{T}}$ and $\boldsymbol{u} = (\boldsymbol{\mu}_1^{\mathrm{T}}, \ldots, \boldsymbol{\mu}_n^{\mathrm{T}})^{\mathrm{T}}$, so that $\boldsymbol{y} \sim \mathcal{N}(\boldsymbol{u}, \mathbf{I}_n \otimes \boldsymbol{\Sigma})$, where $\otimes$ denotes the Kronecker product. Then

$$\operatorname{tr}(\boldsymbol{W}\boldsymbol{A}_i) = \operatorname{tr}\left(\sum_{t=1}^{n} \boldsymbol{z}_t \boldsymbol{z}_t^{\mathrm{T}} \boldsymbol{A}_i\right) = \sum_{t=1}^{n} \boldsymbol{z}_t^{\mathrm{T}} \boldsymbol{A}_i \boldsymbol{z}_t = \boldsymbol{y}^{\mathrm{T}} (\mathbf{I}_n \otimes \boldsymbol{A}_i)\, \boldsymbol{y},$$

so $\mathrm{E}[\prod_{i=1}^{k} \operatorname{tr}(\boldsymbol{W}\boldsymbol{A}_i)]$ is the expectation of a product of quadratic forms in $\boldsymbol{y}$, and Propositions 1 and 2 apply with $\boldsymbol{A}_i \mapsto \mathbf{I}_n \otimes \boldsymbol{A}_i$, $\boldsymbol{\Sigma} \mapsto \mathbf{I}_n \otimes \boldsymbol{\Sigma}$, and $\boldsymbol{\mu} \mapsto \boldsymbol{u}$. The τ-terms become

$$\operatorname{tr}\Big((\mathbf{I}_n \otimes \boldsymbol{A}_{\nu_1})(\mathbf{I}_n \otimes \boldsymbol{\Sigma}) \cdots (\mathbf{I}_n \otimes \boldsymbol{A}_{\nu_p})(\mathbf{I}_n \otimes \boldsymbol{\Sigma})\Big) = \operatorname{tr}\Big(\mathbf{I}_n \otimes \boldsymbol{A}_{\nu_1}\boldsymbol{\Sigma} \cdots \boldsymbol{A}_{\nu_p}\boldsymbol{\Sigma}\Big) = n \operatorname{tr}\Big(\boldsymbol{A}_{\nu_1}\boldsymbol{\Sigma} \cdots \boldsymbol{A}_{\nu_p}\boldsymbol{\Sigma}\Big),$$

which is (36). For the θ-terms, write $\boldsymbol{D} = \boldsymbol{A}_{\nu_1}\boldsymbol{\Sigma} \cdots \boldsymbol{\Sigma}\boldsymbol{A}_{\nu_p}$; then, using the block diagonal structure of $\mathbf{I}_n \otimes \boldsymbol{D}$,

$$\boldsymbol{u}^{\mathrm{T}}(\mathbf{I}_n \otimes \boldsymbol{D})\boldsymbol{u} = \sum_{t=1}^{n} \boldsymbol{\mu}_t^{\mathrm{T}} \boldsymbol{D} \boldsymbol{\mu}_t = \sum_{t=1}^{n} \operatorname{tr}\left(\boldsymbol{D}\boldsymbol{\mu}_t\boldsymbol{\mu}_t^{\mathrm{T}}\right) = \operatorname{tr}\left(\boldsymbol{D}\boldsymbol{\Lambda}\right),$$

which is (35). □

Appendix A.11. Proof of Corollary 3

The argument of Corollary 2 applies verbatim to the augmented vectors: with $\hat{\boldsymbol{y}} = (\hat{\boldsymbol{z}}_1^{\mathrm{T}}, \ldots, \hat{\boldsymbol{z}}_n^{\mathrm{T}})^{\mathrm{T}} \sim \mathcal{N}(\hat{\boldsymbol{u}}, \mathbf{I}_n \otimes \hat{\boldsymbol{\Sigma}})$ and $\hat{\boldsymbol{u}} = (\hat{\boldsymbol{\mu}}_1^{\mathrm{T}}, \ldots, \hat{\boldsymbol{\mu}}_n^{\mathrm{T}})^{\mathrm{T}}$,

$$\operatorname{tr}(\boldsymbol{W}\boldsymbol{A}_i) = \sum_{t=1}^{n} \boldsymbol{z}_t^{\mathrm{H}} \boldsymbol{A}_i \boldsymbol{z}_t = \hat{\boldsymbol{y}}^{\mathrm{T}}(\mathbf{I}_n \otimes \hat{\boldsymbol{A}}_i)\hat{\boldsymbol{y}}.$$

By Lemma 5 applied blockwise,

$$\hat{\boldsymbol{u}}^{\mathrm{T}}(\mathbf{I}_n \otimes \hat{\boldsymbol{D}}_{\boldsymbol{\nu}})\hat{\boldsymbol{u}} = \sum_{t=1}^{n} \frac{\Re(\boldsymbol{\mu}_t^{\mathrm{H}} \boldsymbol{D} \boldsymbol{\mu}_t)}{2^{p-1}} = \frac{\Re\left(\operatorname{tr}(\boldsymbol{D}\boldsymbol{\Lambda})\right)}{2^{p-1}}, \qquad \hat{\tau}_{\boldsymbol{\nu}} = \frac{\Re\left(n \operatorname{tr}(\boldsymbol{A}_{\nu_1}\boldsymbol{\Sigma} \cdots \boldsymbol{A}_{\nu_p}\boldsymbol{\Sigma})\right)}{2^{p-1}},$$

where $\hat{\boldsymbol{D}}_{\boldsymbol{\nu}} = \hat{\boldsymbol{A}}_{\nu_1}\hat{\boldsymbol{\Sigma}} \cdots \hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{A}}_{\nu_p}$ and $\boldsymbol{\Lambda} = \sum_{t=1}^{n} \boldsymbol{\mu}_t\boldsymbol{\mu}_t^{\mathrm{H}}$. Collapsing the powers of two exactly as in the proof of Proposition 3 yields the stated formulas. □

Appendix A.12. Proof of Lemma 6

In $\boldsymbol{\pi}$, the integer j appears s_j times, with b_{ij} copies in block $\boldsymbol{\pi}_i$. A set partition $\boldsymbol{\pi}'$ collapsing to $\boldsymbol{\pi}$ is obtained by distributing the s_j labels $\{s_1 + \cdots + s_{j-1} + 1, \ldots, s_1 + \cdots + s_j\}$ representing $\boldsymbol{A}_j$ among the blocks, with exactly b_{ij} of them in block i, for each j. If the blocks of $\boldsymbol{\pi}$ were regarded as distinguishable, the number of such distributions would be $\prod_{j=1}^{k} s_j! / \prod_{i=1}^{r} b_{ij}!$. However, when $\boldsymbol{\pi}$ contains repeated blocks, distributions that differ only by a permutation of identical blocks produce the same (unordered) set partition $\boldsymbol{\pi}'$, and each $\boldsymbol{\pi}'$ arises from exactly $\prod_{i=1}^{r'} h_i!$ distributions. (Note that two distinct blocks of $\boldsymbol{\pi}'$ can never coincide, since the labels are distinct.) Dividing by $\prod_{i=1}^{r'} h_i!$ yields (39). □

Appendix A.13. Proof of Lemma 7

By definition, the canonical θ-words on the content of $\boldsymbol{\nu}$ are the multiset permutations $\boldsymbol{w}$ with $\boldsymbol{w} \preceq \boldsymbol{w}^R$ in lexicographic order. Since lexicographic order is a total order, every non-palindromic $\boldsymbol{w}$ (i.e., $\boldsymbol{w} \neq \boldsymbol{w}^R$) satisfies exactly one of $\boldsymbol{w} \prec \boldsymbol{w}^R$ or $\boldsymbol{w}^R \prec \boldsymbol{w}$, and the map $\boldsymbol{w} \mapsto \boldsymbol{w}^R$ pairs up the non-palindromic permutations. With $N = N(n_1, \ldots, n_p)$ multiset permutations in total, of which P are palindromic, exactly $(N-P)/2$ satisfy $\boldsymbol{w} \succ \boldsymbol{w}^R$, so the number of canonical words is $N - (N-P)/2 = (N+P)/2$.

It remains to count the palindromic permutations. In a palindrome of length n, position l and position $n+1-l$ carry the same entry, so every entry appears an even number of times in the off-center positions; hence at most one multiplicity n_i can be odd (the entry occupying the center when n is odd), and $P = 0$ if two or more n_i are odd. Otherwise, a palindrome is determined by its first $\lfloor n/2 \rfloor$ positions, which contain $\lfloor n_i/2 \rfloor$ copies of the i-th entry, so the number of palindromes is the multinomial coefficient in (42). □

Appendix A.14. Proof of Lemma 8

By (4), $n_\tau(\boldsymbol{\nu})$ is the number of equivalence classes of arrangements of the content of $\boldsymbol{\nu}$ under rotations and reversal, i.e., the number of bracelets with content $(n_1, \ldots, n_p)$, so it suffices to establish (44). We use Pólya's enumeration theorem [22]; see Read [23] for a review. The cycle index of the cyclic group C_n is $Z(C_n) = n^{-1} \sum_{d|n} \phi(d)\, a_d^{n/d}$, and, substituting the content generating function $a_d = x_1^d + \cdots + x_p^d$, the number of necklaces with content $(n_1, \ldots, n_p)$ is the coefficient of $x_1^{n_1} \cdots x_p^{n_p}$ in $Z(C_n)$, which is readily seen to equal $N_{\text{neck}}(n_1, \ldots, n_p)$ in (43): only the divisors d of $\gcd(n_1, \ldots, n_p)$ contribute, and for each such d the coefficient of $\prod_i x_i^{n_i}$ in $(x_1^d + \cdots + x_p^d)^{n/d}$ is the multinomial coefficient $(n/d)!/\prod_i (n_i/d)!$.

For bracelets, the relevant group is the dihedral group D_n, with cycle index (see Riordan [24, p. 150])

$$Z(D_n) = \begin{cases} \dfrac{Z(C_n)}{2} + \dfrac{a_1 a_2^{(n-1)/2}}{2}, & n \text{ odd}, \\[2ex] \dfrac{Z(C_n)}{2} + \dfrac{a_1^2 a_2^{(n-2)/2} + a_2^{n/2}}{4}, & n \text{ even}. \end{cases}$$

The coefficient of $x_1^{n_1} \cdots x_p^{n_p}$ in $Z(C_n)/2$ contributes $N_{\text{neck}}(n_1, \ldots, n_p)/2$ in both cases; the reflection terms contribute as follows.

Case n odd. Here j, the number of odd n_i, is odd, so $j \geq 1$. The coefficient of $\prod_i x_i^{n_i}$ in $\frac{1}{2}(x_1 + \cdots + x_p)(x_1^2 + \cdots + x_p^2)^{(n-1)/2}$ is nonzero only if exactly one n_i is odd ($j = 1$): the odd-multiplicity variable must be the one drawn from the linear factor, and the remaining choices from the quadratic factor number $((n-1)/2)!/\prod_i \lfloor n_i/2 \rfloor!$. The contribution is therefore $((n-j)/2)!/(2\prod_i \lfloor n_i/2 \rfloor!)$ when $j = 1$ and zero when $j \geq 3$.

Case n even. Here j is even. The reflection polynomial $\frac{1}{4}(x_1 + \cdots + x_p)^2(x_1^2 + \cdots + x_p^2)^{(n-2)/2} + \frac{1}{4}(x_1^2 + \cdots + x_p^2)^{n/2}$ has at most two variables with odd exponent in each monomial, so the coefficient vanishes when $j > 2$. When $j = 0$, both summands contribute $(n/2)!/(4\prod_i (n_i/2)!)$ each, for a total of $(n/2)!/(2\prod_i (n_i/2)!)$. When $j = 2$, only the first summand contributes; the two odd-multiplicity variables must come from the squared linear factor, in $2!/(1!\,1!) = 2$ ways, and the remaining choices number $((n-2)/2)!/\prod_i \lfloor n_i/2 \rfloor!$, for a total of $((n-2)/2)!/(2\prod_i \lfloor n_i/2 \rfloor!)$.

In all cases with $j \leq 2$ the reflection contribution equals $((n-j)/2)!/(2\prod_{i=1}^p \lfloor n_i/2 \rfloor!)$, and it vanishes for $j > 2$, which proves (44). □

Appendix A.15. Proof of Lemma 9

Consider a block of $\boldsymbol{\pi}' \in \mathcal{B}_{|s|}$ containing ℓ distinct labels that collapses to the word content $\boldsymbol{\nu}$, in which the integer l appears b_l times. The collapse map sends each of the $\ell!$ arrangements of the labels to an arrangement of the content of $\boldsymbol{\nu}$; since the labels carrying a common index l can be permuted in $b_l!$ ways without changing the image, the map is uniformly $(\prod_l b_l!)$-to-one from label arrangements onto the $N = \ell!/\prod_l b_l!$ content arrangements.

θ-multiplicities. Assume $\ell \geq 2$ (the case $\ell = 1$ is trivial). The canonical θ-words on the labels are the $\ell!/2$ unordered pairs $\{\boldsymbol{w}, \boldsymbol{w}^R\}$ (no label arrangement is palindromic, the labels being distinct). Such a pair collapses to the term $\theta_{\boldsymbol{p}_j(\boldsymbol{\nu})}$ if and only if the image of $\boldsymbol{w}$ lies in $\{\boldsymbol{p}_j, \boldsymbol{p}_j^R\}$. If $\boldsymbol{p}_j \neq \boldsymbol{p}_j^R$, the preimage of $\{\boldsymbol{p}_j, \boldsymbol{p}_j^R\}$ consists of $2\prod_l b_l!$ label arrangements, forming $\prod_l b_l!$ pairs; if $\boldsymbol{p}_j = \boldsymbol{p}_j^R$ (palindromic), the preimage consists of $\prod_l b_l!$ arrangements, forming $\frac{1}{2}\prod_l b_l!$ pairs. This proves (46).

τ-multiplicities. Assume $\ell \geq 3$ (the cases $\ell \leq 2$ are immediate). The canonical τ-words on the labels are the $(\ell-1)!/2$ orbits of label arrangements under the dihedral group, each orbit of size 2ℓ (proof of Lemma 1). Such an orbit collapses to $\tau_{\boldsymbol{q}_j(\boldsymbol{v})}$ if and only if the image of any of its members lies in the dihedral orbit of $\boldsymbol{q}_j$ as a sequence. If $\boldsymbol{q}_j$ has primitive period ρ_j, its rotations comprise ρ_j distinct sequences; the reflected rotations comprise the same set of sequences if $\boldsymbol{q}_j$ maps to one necklace, and ρ_j further distinct sequences if it maps to two. Hence the dihedral orbit of $\boldsymbol{q}_j$ contains ρ_j or $2\rho_j$ distinct sequences in the two cases, its preimage contains $\rho_j \prod_l b_l!$ or $2\rho_j \prod_l b_l!$ label arrangements, and the number of label orbits in the preimage is obtained by dividing by 2ℓ, which proves (47).

Finally, since every canonical label word collapses to exactly one canonical content word, the multiplicities sum to the total counts: $\sum_j c_j = \ell!/2 = n_\theta(\ell)$ and $\sum_j d_j = (\ell-1)!/2 = n_\tau(\ell)$ for $\ell \geq 3$, with the obvious modifications for small ℓ. □

Appendix A.16. Proof of Proposition 4

By (38) and Proposition 2 applied to the expanded list $\tilde{\boldsymbol{A}}_1, \ldots, \tilde{\boldsymbol{A}}_{|s|}$,

$$\mathrm{E}\left[\prod_{i=1}^{k}(\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}_i\boldsymbol{z})^{s_i}\right] = \mathrm{E}\left[\prod_{i=1}^{|s|}\boldsymbol{z}^{\mathrm{T}}\tilde{\boldsymbol{A}}_i\boldsymbol{z}\right] = \sum_{\boldsymbol{\pi}'\in\mathcal{B}_{|s|}} 2^{|s|-f_1-f_2}\prod_{i=1}^{\ell(\boldsymbol{\pi}')}\tilde{\lambda}_{\boldsymbol{\pi}'_i}, \tag{A.7}$$

where $\tilde{\lambda}_{\boldsymbol{\pi}'_i}$ denotes the block sum (22) evaluated with the matrices $\tilde{\boldsymbol{A}}_l$, $l \in \boldsymbol{\pi}'_i$. Group the set partitions $\boldsymbol{\pi}'$ according to the multiset partition $\boldsymbol{\pi} \in \mathcal{B}_{\boldsymbol{s}}$ to which they collapse. All $\boldsymbol{\pi}'$ in the group of $\boldsymbol{\pi}$ share the same block-size profile, hence the same f_1 and f_2, and by Lemma 6 the group contains $c_{\boldsymbol{s}}(\boldsymbol{\pi})$ set partitions. Moreover, for each block, expressing $\tilde{\lambda}_{\boldsymbol{\pi}'_i}$ in terms of the original matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ amounts to collapsing each canonical label word to its content word; by Lemma 9, the $n_\theta(\ell)$ canonical θ-words collapse onto the canonical words $\boldsymbol{p}_j(\boldsymbol{\pi}_i)$ with multiplicities c_j, and the $n_\tau(\ell)$ canonical τ-words collapse onto $\boldsymbol{q}_j(\boldsymbol{\pi}_i)$ with multiplicities d_j, so $\tilde{\lambda}_{\boldsymbol{\pi}'_i} = \lambda_{\boldsymbol{\pi}_i}$ with $\lambda_{\boldsymbol{\pi}_i}$ as defined in (48). In particular, $\tilde{\lambda}_{\boldsymbol{\pi}'_i}$ depends on $\boldsymbol{\pi}'$ only through $\boldsymbol{\pi}$, and summing (A.7) over each group yields (49).

For the complex case, the identical grouping applied to (31) of Proposition 3 gives the same formula with $2^{|s|-f_1-f_2}$ replaced by $2^{\ell(\boldsymbol{\pi})-f_1-f_2}$ (the number of blocks is also constant within each group) and with the real parts taken in the θ- and τ-terms.

Finally, we verify the count (45) of distinct terms. Expanding the product $\prod_{i=1}^{\ell(\boldsymbol{\pi})}\lambda_{\boldsymbol{\pi}_i}$ in (49), a term is obtained by selecting one canonical word (with its designation) from each block; selections that differ only by a permutation of identical blocks produce the same term, so the number of distinct terms produced by $\boldsymbol{\pi}$ is $\prod_{i=1}^{r'}\binom{n(\tilde{\boldsymbol{\pi}}_i)+h_i-1}{h_i}$, the number of ways of choosing, for each family of h_i identical blocks, an unordered selection with repetition of h_i canonical terms from the $n(\tilde{\boldsymbol{\pi}}_i)$ available ones. Terms produced by different $\boldsymbol{\pi} \in \mathcal{B}_{\boldsymbol{s}}$ are distinct, because the multiset of contents of the index words of a term recovers $\boldsymbol{\pi}$. Summing over $\boldsymbol{\pi}$ gives (45). □

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