On the Moments of Ratios of Quadratic Forms in Normal Random Variables*

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Abstract

In this paper, we present both integral and infinite series expressions of $\mu_q^p \equiv E[(x'Ax)^p/(x'Bx)^q]$ when $x \sim N(\mu, I_n)$, where p, q are nonnegative real numbers, A is a symmetric matrix, and B is a positive semi-definite matrix. We derive a double infinite series expansion of μ_q^p (a single series when p is an integer) directly from the integral representation, together with efficient numerical methods for its evaluation, and we show that the classical triple infinite series expansion of Smith (1989) in terms of top-order invariant polynomials follows from it as a corollary and remains valid whenever μ_q^p exists, in particular when A or B is positive semi-definite.

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1. Introduction

Let $x = [x_1, \ldots, x_n]' \sim N(\mu, I_n)$ be a normal random vector, A be a symmetric nonzero matrix, and B be a positive semi-definite matrix. For nonnegative real numbers p and q, we are interested in obtaining computationally efficient expressions of

$$\mu_q^p \equiv E\left[\frac{(x'Ax)^p}{(x'Bx)^q}\right].$$

When p is not an integer, we assume that A is also a positive semi-definite matrix in order for $(x'Ax)^p$ to be well defined.[1]

Since many statistical estimators can be written as ratios of quadratic forms, the computation of μ_q^p has been of great interest to statisticians and econometricians, especially for $p = q$. Broadly speaking, there are two different approaches for evaluating μ_q^p. The first approach is an integration approach that starts with Sawa [11], who provides an integral formula that allows us to compute μ_q^p when p is an integer. This method is by far the most popular one in the literature. Magnus [6] provides a nice discussion, in particular the numerical aspect, of this method, and Meng [8] provides an excellent review of this literature. When p is not an integer, the integral formula for μ_q^p is also available (see, for example, Mathai and Provost [7] (Section 4.5d) and Meng [8] (Lemma 2)), but this would typically involve a double integral. In addition, the integrand is often difficult to evaluate, so this formula has seen little use in practice.

The second approach is to rely on some infinite series expansion of μ_q^p. Smith [12] is the first one to provide a triple infinite series (or double infinite series if p is an integer) expansion of μ_q^p in terms of top-order invariant polynomials. Due to the difficulty of computing top-order invariant polynomials and the complexity of this approach, his formula of μ_q^p appears to be only of academic interest. Recently, Hillier, Kan, and Wang [4] provide an efficient method for

[1] Our results can be easily adapted to deal with the case when $x \sim N(\mu, \Sigma)$, where Σ is a positive definite matrix. This is because we can write $x'Ax = \tilde{x}'(\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}})\tilde{x}$ and $x'Bx = \tilde{x}'(\Sigma^{\frac{1}{2}}B\Sigma^{\frac{1}{2}})\tilde{x}$, where $\tilde{x} = \Sigma^{-\frac{1}{2}}x \sim N(\Sigma^{-\frac{1}{2}}\mu, I_n)$. In addition, we can also deal with the case that B is a negative semi-definite matrix if q is an integer.

computing top-order invariant polynomials. Nevertheless, they are able to implement this formula only for integral p and $\mu = 0_n$. Note that in deriving the infinite series expansion of μ_q^p, Smith [12] needs to assume that both A and B are positive definite. Therefore, it appears that his formula is not applicable for the case when A or B is positive semi-definite.

The objective of this paper is to improve both approaches for evaluating μ_q^p. For the integration approach, we provide an explicit expression of the integrand when p is not an integer. In addition, we provide a numerically efficient method for computing the integrand. This improvement is particularly important in light of the fact that this approach calls for a double integral. With our numerical method, it becomes practical to use the double integral formula for evaluating μ_q^p. For the infinite series approach, we proceed in the reverse order of the earlier literature. We first derive, directly from the integral representation, a double infinite series expansion of μ_q^p, which collapses to a single infinite series when p is an integer. This expansion was introduced by Hillier, Kan, and Wang [5] for the case of integral p and positive definite B; we extend it to non-integral p and to positive semi-definite A and B, and we equip it with a fast recurrence algorithm for its coefficients, which makes the infinite series approach very competitive against the integration approach. We then show that Smith's triple infinite series expansion of μ_q^p in terms of top-order invariant polynomials, which was originally established under the assumption that both A and B are positive definite, follows from our double series by a rearrangement, and therefore continues to hold whenever μ_q^p exists, also when A or B is positive semi-definite. Smith [12] attempted to deal with some special semi-definite cases but left the general case as an open question, which our result settles. Compared with the original development in the published version of this paper, the present one has three advantages: the series that is actually used for computation is the one that is proved first; the convergence analysis is carried out only once, for series of nonnegative terms, so that the various interchanges of summation and integration are immediate consequences of Tonelli's theorem; and the case of integral p with $\tilde{A}_{12} \neq 0$ (in the notation of Section 2), which was not covered by the original argument, is now included.

The rest of the paper is organized as follows. Section 2 presents the necessary and sufficient conditions for the existence of μ_q^p. Section 3 presents an integral expression of μ_q^p and discusses a numerically efficient approach of evaluating the integrand. In Section 4, we develop the double infinite series expression of μ_q^p, provide a numerically efficient approach for evaluating its coefficients, and discuss two numerical safeguards that are important in practice. Section 5 shows that Smith's triple infinite series expression of μ_q^p in terms of top-order invariant polynomials is a corollary of the results in Section 4, and that it continues to hold even when A or B is positive semi-definite. Section 6 concludes the paper. Proofs of all the propositions are collected in the Appendix.

2. Necessary and sufficient conditions for the existence of μ_q^p

Before we present different expressions of μ_q^p, we need to first understand the conditions for the existence of μ_q^p. Suppose the rank of B is $m \leq n$. Let $P_1 D_{1b} P_1' = B$, where $D_{1b} = \mathrm{Diag}(b_1, \ldots, b_m)$ with $b_1 \geq \cdots \geq b_m > 0$ are the positive eigenvalues of B, and P_1 is an $n \times m$ matrix of the eigenvectors associated with the positive eigenvalues of B. Denoting $P = [P_1,\ P_2]$, where P_2 is an $n \times (n-m)$ matrix of the eigenvectors associated with the zero eigenvalues of B, we can write

$$\frac{(x'Ax)^p}{(x'Bx)^q} = \frac{(x'PP'APP'x)^p}{(x'P_1D_{1b}P_1'x)^q} = \frac{(z'\tilde{A}z)^p}{(z_1'D_{1b}z_1)^q},$$

where

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P_1'x \\ P_2'x \end{bmatrix} \sim N(P'\mu, I_n),$$

and

$$\tilde{A} = P'AP = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

with $\tilde{A}_{ij} = P_i'AP_j$. The following proposition presents the necessary and sufficient conditions for the existence of μ_q^p.

Proposition 1. *(1) Suppose B is a positive definite matrix. μ_q^p exists if and only if $\frac{n}{2} + p > q$.*

(2) Suppose B is a positive semi-definite matrix with rank $m < n$. We have three cases to consider. (i) When $\tilde{A}_{12} = 0_{m\times(n-m)}$ and $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$, μ_q^p exists if and only if $\frac{m}{2} + p > q$. (ii) When $\tilde{A}_{12} \neq 0_{m\times(n-m)}$ and $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$, μ_q^p exists if and only if $\frac{m+p}{2} > q$. (iii) When $\tilde{A}_{22} \neq 0_{(n-m)\times(n-m)}$, μ_q^p exists if and only if $\frac{m}{2} > q$.

Roberts [9] (Sections 3.1 and 7.2.2) has already shown that the conditions in Proposition 1 are sufficient for the existence of μ_q^p. He also shows that when p is an integer, the conditions in Proposition 1 are both necessary and sufficient. However, our Proposition 1 provides a stronger result by showing that the conditions are necessary and sufficient even when p is not an integer. Note that when p is not an integer, we need to assume A is positive semi-definite. It can be easily shown that when A is positive semi-definite, $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$ implies $\tilde{A}_{12} = 0_{m\times(n-m)}$,[2] so we can eliminate case (ii) from consideration when A is positive semi-definite.

3. Integral expression of $\boldsymbol{\mu_q^p}$

When p is an integer, the most popular method for numerical evaluation of μ_q^p is to use the results of Sawa [11] and Cressie, Davis, Folks, and Policello [3] to write

$$\mu_q^p = \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} \left. \frac{\partial^p}{\partial t_1^p} \phi(t_1, t_2) \right|_{t_1=0, t_2=-t} \mathrm{d}t, \tag{1}$$

where

$$\phi(t_1, t_2) = |I_n - 2t_1 A - 2t_2 B|^{-\frac{1}{2}} \exp\left(\frac{\mu'(I_n - 2t_1 A - 2t_2 B)^{-1}\mu}{2} - \frac{\mu'\mu}{2} \right)$$

is the joint moment generating function of $x'Ax$ and $x'Bx$. Meng [8] provides an excellent review of this literature.

When p is not an integer, Mathai and Provost [7] (Eq.4.5d.8) show that an integration formula is also available for μ_q^p (see also Lemma 2 of Meng [8]), and it is given by the following double integral expression:

$$\mu_q^p = \frac{1}{\Gamma(\langle p \rangle)\Gamma(q)} \int_0^\infty \int_0^\infty s^{\langle p \rangle - 1} t^{q-1} \left. \frac{\partial^{\lceil p \rceil}}{\partial t_1^{\lceil p \rceil}} \phi(t_1, t_2) \right|_{t_1=-s, t_2=-t} \mathrm{d}s \mathrm{d}t, \tag{2}$$

where $\lceil p \rceil$ is the smallest integer that is greater than or equal to p, and $\langle p \rangle = \lceil p \rceil - p$.

[2] Since A is positive semi-definite, $\tilde{A}$ is also positive semi-definite. For a positive semi-definite matrix $\tilde{A} = (\tilde{a}_{ij})$, we have $\tilde{a}_{ii}\tilde{a}_{jj} \geq \tilde{a}_{ij}^2$ for $j \neq i$. Therefore, $\tilde{a}_{ii} = 0$ implies $\tilde{a}_{ij} = 0$ for $j \neq i$. This suggests that if a diagonal element of a positive semi-definite matrix is zero, then the entire row and column to which it belongs must also be zero.

In order for us to use (1) or (2) to evaluate μ_q^p, we need an explicit expression for the partial derivatives of $\phi(t_1, t_2)$. Magnus [6] presents a simple expression of this for the case when p is an integer. In the following proposition, we present a slightly different expression of his result as well as a corresponding expression for the case when p is not an integer.

Proposition 2. *Suppose μ_q^p exists. When p is an integer, we have*

$$\mu_q^p = \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} \phi(0, -t) E[(w'Rw)^p] \mathrm{d}t, \tag{3}$$

where $R = L'AL$, $w \sim N(\tilde{\mu}, I_n)$ with $\tilde{\mu} = L'\mu$, and L is an $n \times n$ matrix such that $LL' = (I_n + 2tB)^{-1}$. When p is not an integer, we have

$$\mu_q^p = \frac{1}{\Gamma(\langle p \rangle)\Gamma(q)} \int_0^\infty \int_0^\infty s^{\langle p \rangle - 1} t^{q-1} \phi(-s, -t) E[(w'Rw)^{\lceil p \rceil}] \mathrm{d}s \mathrm{d}t, \tag{4}$$

but L is now defined as an $n \times n$ matrix such that $LL' = (I_n + 2sA + 2tB)^{-1}$.

There are two computational issues for us to tackle when using (4) to compute μ_q^p. The first issue is the computation of $E[(w'Rw)^k]$. When k is small, a simple explicit formula of $E[(w'Rw)^k]$ is available. For example, when $k = 1$, we have $E[w'Rw] = \mathrm{tr}(R) + \tilde{\mu}'R\tilde{\mu}$. However, when k is large, it is very tedious and computationally expensive to use the explicit formula (see Magnus [6] for an explicit formula of $E[(w'Rw)^k]$). In the following, we provide a fast recurrence algorithm for computing $E[(w'Rw)^k]$.

Let

$$\tilde{D}(t) = |I_n - tR|^{-\frac{1}{2}} \exp\left(\frac{\tilde{\mu}'(I_n - tR)^{-1}\tilde{\mu} - \tilde{\mu}'\tilde{\mu}}{2} \right) = \sum_{k=0}^\infty \tilde{d}_k t^k.$$

It can be readily shown that $\tilde{d}_k = E[(w'Rw)^k]/(2^k k!)$ and we have the following recurrence relation (see, for example, Ruben [10] and Mathai and Provost [7] (Eq.3.2b.8))

$$\tilde{d}_k = \frac{1}{2k} \sum_{j=1}^k p_j \tilde{d}_{k-j}, \tag{5}$$

where

$$p_j = \mathrm{tr}(R^j) + j\tilde{\mu}'R^j\tilde{\mu}.$$

Let $Q\Lambda Q' = R$, where $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix of the eigenvalues of R, and $Q = [q_1, \ldots, q_n]$ is a matrix of the corresponding eigenvectors. We can then write

$$p_j = \sum_{i=1}^{n} (\lambda_i^j + j\delta_i \lambda_i^j), \tag{6}$$

where $\delta_i = (q_i'\tilde{\mu})^2$. Using the initial condition $\tilde{d}_0 = 1$, we can use (5) successively to obtain $\tilde{d}_k$. While (5) is far more efficient than the explicit formula for computing $\tilde{d}_k$, it has a shortcoming in that the length of recursion increases with k, so it can be inefficient when k is large.

Using a method suggested by Brown [1], it is possible to update $\tilde{d}_k$ from $\tilde{d}_{k-1}$ without the need of using all the $\tilde{d}_i$ for $0 \le i \le k-1$. In order to obtain this short recurrence relation, we substitute (6) into (5) and exchange the order of summation to obtain the following expression

$$\tilde{d}_k = \frac{1}{2k} \sum_{i=1}^{n} \sum_{j=1}^{k} (\lambda_i^j \tilde{d}_{k-j} + j\delta_i \lambda_i^j \tilde{d}_{k-j}) = \frac{1}{2k} \sum_{i=1}^{n} (u_{i,k} + v_{i,k}), \tag{7}$$

where[3]

$$u_{i,k} = \sum_{j=1}^{k} \lambda_i^j \tilde{d}_{k-j}, \qquad v_{i,k} = \delta_i \sum_{j=1}^{k} j\lambda_i^j \tilde{d}_{k-j}. \tag{8}$$

It can be easily verified that $u_{i,k}$ and $v_{i,k}$ can be updated using the following recurrence relation:

$$u_{i,k} = \lambda_i (u_{i,k-1} + \tilde{d}_{k-1}), \qquad v_{i,k} = \delta_i u_{i,k} + \lambda_i v_{i,k-1}, \tag{9}$$

with the initial conditions $u_{i,0} = 0$ and $v_{i,0} = 0$. This method is extremely efficient because we update $u_{i,k}$ and $v_{i,k}$ using just $u_{i,k-1}$, $v_{i,k-1}$ and $\tilde{d}_{k-1}$. As a result, the memory requirement for this recurrence relation is only $2n + 1$ elements (n elements for $u_{i,k}$, n elements for $v_{i,k}$ and one element for $\tilde{d}_k$) regardless of k, and the computation time for updating $\tilde{d}_k$ does not increase with k.

The second issue is the computational efficiency of the integrand. In general, R depends on both s and t, and since we need to obtain the eigenvalues

[3] When $\mu = 0_n$, we can drop the $v_{i,k}$ terms because they are equal to zero.

and eigenvectors of R in order to compute the moments of $w'Rw$, this can be extremely time consuming. In order to speed up the computation of the integrand, we propose a method that frees us from performing eigenvalue decomposition of R. Under this method, eigenvalue decomposition is only performed at the outer integral, and there is no need to perform eigenvalue decomposition of R as we change the value of s in the inner integral.

Our method hinges on choosing an appropriate L such that $R = L'AL$ is a diagonal matrix. Let $PD_bP' = B$, where $D_b = \mathrm{Diag}(b_1, \ldots, b_n)$ is a diagonal matrix of the eigenvalues of B (some of the b_i's can be zero), and P is an $n \times n$ matrix of the corresponding eigenvectors. Denoting $\tilde{A} = P'AP$, we can then write

$$I_n + 2sA + 2tB = P(I_n + 2s\tilde{A} + 2tD_b)P' = P\Delta^{-1}(I_n + 2s\breve{A})\Delta^{-1}P',$$

where $\Delta = (I_n + 2tD_b)^{-\frac{1}{2}}$ and $\breve{A} = \Delta\tilde{A}\Delta$. Let $HD_{\breve{a}}H' = \breve{A}$, where $D_{\breve{a}} = \mathrm{Diag}(\breve{a}_1, \ldots, \breve{a}_n)$ is a diagonal matrix of the eigenvalues of $\breve{A}$, and H is an $n \times n$ matrix of the corresponding eigenvectors. With these transformations, we can write

$$(I_n + 2sA + 2tB)^{-1} = P\Delta H(I_n + 2sD_{\breve{a}})^{-1}H'\Delta P',$$

and choose $L = P\Delta H(I_n + 2sD_{\breve{a}})^{-\frac{1}{2}}$. Under this choice of L, we have

$$\begin{aligned} R &= L'AL \\ &= (I_n + 2sD_{\breve{a}})^{-\frac{1}{2}}H'\Delta P'AP\Delta H(I_n + 2sD_{\breve{a}})^{-\frac{1}{2}} \\ &= (I_n + 2sD_{\breve{a}})^{-\frac{1}{2}}H'\breve{A}H(I_n + 2sD_{\breve{a}})^{-\frac{1}{2}} \\ &= (I_n + 2sD_{\breve{a}})^{-\frac{1}{2}}D_{\breve{a}}(I_n + 2sD_{\breve{a}})^{-\frac{1}{2}}, \end{aligned}$$

which is a diagonal matrix with diagonal elements $\lambda_i = \breve{a}_i/(1 + 2s\breve{a}_i)$. In addition, we have $\delta_i = \tilde{\mu}_i^2$, where

$$\tilde{\mu} = L'\mu = (I_n + 2sD_{\breve{a}})^{-\frac{1}{2}}H'\Delta P'\mu.$$

The important point to note here is that we can obtain λ_i and δ_i by using H and $D_{\breve{a}}$, both of which depend on t but not on s, so we no longer need to perform eigenvalue decomposition of R in the inner integral. In addition, we have

$$|I_n + 2sA + 2tB|^{-\frac{1}{2}} = |\Delta||I_n + 2sD_{\breve{a}}|^{-\frac{1}{2}} = \prod_{i=1}^{n} \frac{1}{(1 + 2tb_i)^{\frac{1}{2}}} \prod_{i=1}^{n} \frac{1}{(1 + 2s\breve{a}_i)^{\frac{1}{2}}}.$$

With all these expressions, we can rewrite (4) as

$$\mu_q^p = \frac{e^{-\frac{\mu'\mu}{2}} 2^{\lceil p \rceil} \lceil p \rceil!}{\Gamma(\langle p \rangle)\Gamma(q)} \int_0^\infty \frac{t^{q-1}}{\prod_{i=1}^n (1+2tb_i)^{\frac{1}{2}}} \left[\int_0^\infty \frac{s^{\langle p \rangle - 1} e^{\frac{\tilde{\mu}'\tilde{\mu}}{2}} \tilde{d}_{\lceil p \rceil}}{\prod_{i=1}^n (1+2s\breve{a}_i)^{\frac{1}{2}}} \mathrm{d}s \right] \mathrm{d}t,$$

and this is our preferred integral expression for evaluating μ_q^p when p is not an integer.[4]

4. Double infinite series expression of μ_q^p

In this section, we present an infinite series expansion of μ_q^p in terms of a double infinite series, which reduces to a single infinite series when p is an integer. For the case that p is an integer and B is positive definite, this expansion was proposed by Hillier, Kan, and Wang [5]. We extend it to non-integral p and to positive semi-definite A and B, and we derive it directly from the integral representation of Section 3, which also provides a unified framework that allows us to reconcile the two different approaches (integral vs. infinite series) of evaluating μ_q^p. In order to present our results, we first define two generating functions

$$\begin{aligned} H(t_1,t_2) &= |I_n - t_1A_1 - t_2A_2|^{-\frac{1}{2}} \exp\left(\frac{(1-t_1-t_2)\mu'(I_n - t_1A_1 - t_2A_2)^{-1}\mu}{2} - \frac{\mu'\mu}{2}\right) \\ &= \sum_{i=0}^\infty \sum_{j=0}^\infty h_{i,j}(A_1,A_2) t_1^i t_2^j, \\ \tilde{H}(t_1,t_2) &= |I_n - t_1A_1 - t_2A_2|^{-\frac{1}{2}} \exp\left(\frac{(1-t_2)\mu'(I_n - t_1A_1 - t_2A_2)^{-1}\mu}{2} - \frac{\mu'\mu}{2}\right) \\ &= \sum_{i=0}^\infty \sum_{j=0}^\infty \tilde{h}_{i,j}(A_1,A_2) t_1^i t_2^j, \end{aligned}$$

where A_1 and A_2 are two symmetric matrices. With $h_{i,j}$ and $\tilde{h}_{i,j}$ defined, the following proposition presents our new infinite series expression of μ_q^p.

[4] A set of Matlab programs for computing μ_q^p using this integration method and other methods discussed later in the paper is available at `https://www-2.rotman.utoronto.ca/~kan/research.htm`.

Proposition 3. *Suppose μ_q^p exists. When p is not an integer, we have*

$$\begin{aligned}
\mu_q^p &= \frac{2^{p-q}\beta^q\Gamma\left(\frac{n}{2}+p-q\right)}{\alpha^p}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-p)_i(q)_j}{\Gamma\left(\frac{n}{2}+i+j\right)}h_{i,j}(\hat{A},\hat{B}) \\
&= \frac{2^{p-q}\beta^q\Gamma\left(\frac{n}{2}+p-q\right)}{\alpha^p}\sum_{k=0}^{\infty}\frac{w_k}{\Gamma\left(\frac{n}{2}+k\right)}, \qquad (10)
\end{aligned}$$

where $w_k = \sum_{i=0}^{k}(-p)_i(q)_{k-i}h_{i,k-i}(\hat{A},\hat{B})$, $\hat{A} = I_n - \alpha A$, $\hat{B} = I_n - \beta B$ with $0 < \alpha \leq 1/a_{\max}$, $0 < \beta \leq 1/b_{\max}$, and $a_{\max}$ and $b_{\max}$ are the maximum eigenvalues of A and B, respectively. When p is an integer, we have

$$\mu_q^p = \frac{2^{p-q}\beta^q p!\Gamma\left(\frac{n}{2}+p-q\right)}{\Gamma\left(\frac{n}{2}+p\right)}\sum_{j=0}^{\infty}\frac{(q)_j}{\left(\frac{n}{2}+p\right)_j}\tilde{h}_{p,j}(A,\hat{B}). \qquad (11)$$

Note that when $\mu = 0_n$, $h_{i,j}(\hat{A},\hat{B}) = d_{i,j}(\hat{A},\hat{B})$ and $\tilde{h}_{i,j}(A,\hat{B}) = d_{i,j}(A,\hat{B})$. In order to use (10) and (11), we need to have an algorithm to compute $h_{i,j}$ and $\tilde{h}_{i,j}$. The following proposition shows that both $h_{i,j}$ and $\tilde{h}_{i,j}$ can be obtained recursively, with the initial conditions $h_{0,0} = 1$ and $\tilde{h}_{0,0} = 1$.

Proposition 4. *Defining $\tau_{i,j} = [t_1^i t_2^j]\mathrm{tr}((t_1A_1+t_2A_2)^{i+j})$ and $\eta_{i,j} = [t_1^i t_2^j]\mu'(t_1A_1+t_2A_2)^{i+j}\mu$, we have*

$$\begin{aligned}
h_{i,j} &= \frac{1}{2(i+j)}\sum_{\nu_1=0}^{i}\sum_{\substack{\nu_2=0\\ \nu_1+\nu_2>0}}^{j}p_{\nu_1,\nu_2}h_{i-\nu_1,j-\nu_2}, \\
\tilde{h}_{i,j} &= \frac{1}{2(i+j)}\sum_{\nu_1=0}^{i}\sum_{\substack{\nu_2=0\\ \nu_1+\nu_2>0}}^{j}\tilde{p}_{\nu_1,\nu_2}\tilde{h}_{i-\nu_1,j-\nu_2}
\end{aligned}$$

where

$$\begin{aligned}
p_{\nu_1,\nu_2} &= \tau_{\nu_1,\nu_2} + (\nu_1+\nu_2)(\eta_{\nu_1,\nu_2} - \eta_{\nu_1-1,\nu_2} - \eta_{\nu_1,\nu_2-1}), \\
\tilde{p}_{\nu_1,\nu_2} &= \tau_{\nu_1,\nu_2} + (\nu_1+\nu_2)(\eta_{\nu_1,\nu_2} - \eta_{\nu_1,\nu_2-1}),
\end{aligned}$$

with the convention that $\eta_{i,j} = 0$ when i or j is negative.

Remark 1. *When p is not an integer, A is required to be positive semi-definite, and the restrictions $0 < \alpha \leq 1/a_{\max}$ and $0 < \beta \leq 1/b_{\max}$ then* *guarantee that all the eigenvalues of $\hat{A}$ and $\hat{B}$ lie in $[0,1]$, a fact that our proofs exploit (it makes all the coefficients in the relevant generating function expansions nonnegative). This entails no loss for computational purposes: the natural choice in practice, and the one made in our programs, is $\alpha = 1/a_{\max}$ and $\beta = 1/b_{\max}$. When both A and B are positive definite, (10) in fact holds for the wider ranges $0 < \alpha < 2/a_{\max}$ and $0 < \beta < 2/b_{\max}$ considered by Smith [12]. Note also that (11) involves A itself rather than $\hat{A}$, so no restriction on A (other than symmetry) or on α is needed there.*

There are two major problems with using the recursions in Proposition 4. First, it is computationally expensive. For instance, $h_{i,j}$ is expressed as a linear combination of all the h_{ν_1,ν_2}'s with $\nu_1 \le i$ and $\nu_2 \le j$, so this requires a lot of storage space, and is computationally expensive when i and j are large. Second, $\tau_{i,j}$ and $\eta_{i,j}$ are not easy to compute because a binomial expansion on $(t_1A_1 + t_2A_2)^{i+j}$ can lead to many terms when $i+j$ is large.

In order to overcome these two problems, we present an efficient algorithm for computing $h_{i,j}$ and $\tilde{h}_{i,j}$, which can be regarded as a multivariate generalization of Brown's [1] algorithm, and is introduced in Hillier, Kan, and Wang [5]. To present this algorithm, we let $A(\boldsymbol{t}) = t_1A_1 + t_2A_2$ and define matrix $G_{i,k-i}$ and vector $g_{i,k-i}$ as follows:

$$\begin{aligned} G_{i,k-i} &= \sum_{\nu_1=0}^{i} \sum_{\substack{\nu_2=0 \\ \nu_1+\nu_2>0}}^{k-i} [t_1^{\nu_1} t_2^{\nu_2}] A(\boldsymbol{t})^{\nu_1+\nu_2} h_{i-\nu_1,k-i-\nu_2}, \\ g_{i,k-i} &= \sum_{\nu_1=0}^{i} \sum_{\substack{\nu_2=0 \\ \nu_1+\nu_2>0}}^{k-i} (\nu_1+\nu_2) \left([t_1^{\nu_1} t_2^{\nu_2}] A(\boldsymbol{t})^{\nu_1+\nu_2} - [t_1^{\nu_1} t_2^{\nu_2-1}] A(\boldsymbol{t})^{\nu_1+\nu_2-1} \right. \\ &\qquad \left. - [t_1^{\nu_1-1} t_2^{\nu_2}] A(\boldsymbol{t})^{\nu_1+\nu_2-1} \right) \mu h_{i-\nu_1,k-i-\nu_2}. \end{aligned}$$

With $G_{i,k-i}$ and $g_{i,k-i}$ available, we can then obtain $h_{i,k-i}$ using

$$h_{i,k-i} = \frac{1}{2k} \sum_{\nu_1=0}^{i} \sum_{\substack{\nu_2=0 \\ \nu_1+\nu_2>0}}^{k-i} p_{\nu_1,\nu_2} h_{i-\nu_1,k-i-\nu_2} = \frac{\operatorname{tr}(G_{i,k-i}) + \mu' g_{i,k-i}}{2k}.$$

As it turns out, both $G_{i,k-i}$ and $g_{i,k-i}$ can be obtained by short recursions, which are given by

$$
\begin{aligned}
G_{i,k-i} &= A_1(h_{i-1,k-i}I_n + G_{i-1,k-i}) + A_2(h_{i,k-1-i}I_n + G_{i,k-1-i}),\\
g_{i,k-i} &= (G_{i,k-i} - G_{i,k-1-i} - G_{i-1,k-i})\mu - (h_{i,k-1-i} + h_{i-1,k-i})\mu\\
&\quad + A_1 g_{i-1,k-i} + A_2 g_{i,k-1-i}.
\end{aligned}
$$

Here, we adopt the convention that $G_{i,j} = 0_{n\times n}$, $g_{i,j} = 0_n$, $h_{i,j} = 0$ when either i or j is negative.

Together with the initial conditions $h_{0,0} = 1$, $g_{0,0} = 0_n$, and $G_{0,0} = 0_{n\times n}$, the above two recursions allow us to efficiently compute $h_{i,k-i}$ for $i = 0,\dots,k$ using only $h_{i,k-1-i}$, $G_{i,k-1-i}$ and $g_{i,k-1-i}$ for $i = 0,\dots,k-1$. This stands in contrast to the long recursion in Proposition 4. More importantly, the updating time of $h_{i,k-i}$ is the same regardless of k, and this is particularly desirable for the computation of μ_q^p that may require summing up $h_{i,k-i}$ for very large values of k.

With our new infinite series representation of μ_q^p and the corresponding fast updating algorithm for $h_{i,k-i}$, the infinite series approach becomes very competitive against the integration approach for numerical evaluation of μ_q^p. For example, with $n = 120$, we find that numerical calculation of μ_q^p based on the new infinite series representation with the fast algorithm is on average ten times faster than using the integration formula in Section 3.

An important advantage of (10) over the integration approach is that $h_{i,j}(\hat{A},\hat{B})$ in (10) does not depend on p and q. Therefore, if one needs to compute a table of μ_q^p for different values of p and q, then one only needs to compute $h_{i,j}$'s once and they can be used to update the series approximation of μ_q^p for all values of p and q. In contrast, the integration approach needs to perform a separate integration for each single p and q, and it can be much more time consuming.

For the sake of completeness, we also present the fast updating algorithm for $\tilde{h}_{i,k-i}$. Defining matrix $\tilde{G}_{i,k-i}$ and vector $\tilde{g}_{i,k-i}$ as follows:

$$
\begin{aligned}
\tilde{G}_{i,k-i} &= \sum_{\nu_1=0}^{i}\sum_{\substack{\nu_2=0\\ \nu_1+\nu_2>0}}^{k-i}[t_1^{\nu_1}t_2^{\nu_2}]A(\boldsymbol{t})^{\nu_1+\nu_2}\tilde{h}_{i-\nu_1,k-i-\nu_2},\\
\tilde{g}_{i,k-i} &= \sum_{\nu_1=0}^{i}\sum_{\substack{\nu_2=0\\ \nu_1+\nu_2>0}}^{k-i}(\nu_1+\nu_2)\left([t_1^{\nu_1}t_2^{\nu_2}]A(\boldsymbol{t})^{\nu_1+\nu_2} - [t_1^{\nu_1}t_2^{\nu_2-1}]A(\boldsymbol{t})^{\nu_1+\nu_2-1}\right)\mu\tilde{h}_{i-\nu_1,k-i-\nu_2},
\end{aligned}
$$

then we have

$$\tilde{h}_{i,k-i} = \frac{\mathrm{tr}(\tilde{G}_{i,k-i}) + \mu'\tilde{g}_{i,k-i}}{2k}.$$

It can be easily verified that $\tilde{G}_{i,k-i}$ and $\tilde{g}_{i,k-i}$ have the following recurrence relations:

$$\begin{aligned}
\tilde{G}_{i,k-i} &= A_1(\tilde{h}_{i-1,k-i}I_n + \tilde{G}_{i-1,k-i}) + A_2(\tilde{h}_{i,k-1-i}I_n + \tilde{G}_{i,k-1-i}), \\
\tilde{g}_{i,k-i} &= (\tilde{G}_{i,k-i} - \tilde{G}_{i,k-1-i})\mu - \tilde{h}_{i,k-1-i}\mu + A_1\tilde{g}_{i-1,k-i} + A_2\tilde{g}_{i,k-1-i},
\end{aligned}$$

with the initial conditions $\tilde{h}_{0,0} = 1$, $\tilde{g}_{0,0} = 0_n$, and $\tilde{G}_{0,0} = 0_{n\times n}$.

We conclude this section with two numerical caveats that are important in practice. First, when B is rank deficient ($m < n$), $\hat{B}$ has unit eigenvalues and the terms of (10) and (11) decay only at a polynomial rate — of the order $k^{-(1+\frac{m}{2}-q)}$ when $\tilde{A}_{22} \neq 0_{(n-m)\times(n-m)}$ and $k^{-(1+\frac{m+p}{2}-q)}$ when $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$ and $\tilde{A}_{12} \neq 0_{m\times(n-m)}$, the exponent being positive precisely when μ_q^p exists. A truncated sum should then be accompanied by a tail estimate: fitting the decay exponent s from the last computed terms and adding $T_J\, J/(s-1)$, where T_J is the last term, reduces the truncation error by several orders of magnitude. Second, the coefficients $h_{i,j}$ and $\tilde{h}_{i,j}$ alternate in sign and can be as large as $e^{\mu'\mu/2}$, whereas μ_q^p itself is smaller than the largest term of the series by a factor of order $e^{-\mu'\mu/2}$. In fixed-precision arithmetic the series therefore loses roughly $\mu'\mu/(2\ln 10)$ significant digits to cancellation, and for $\mu'\mu \gtrsim 60$ (in double precision) the integration approach of Section 3 should be used instead. Both safeguards are implemented in our programs.

5. Smith's expansion as a corollary

Besides the integral and double series expressions of μ_q^p, there is a triple infinite series expression of μ_q^p given by Smith [12] in terms of top-order invariant polynomials. In this section, we show that Smith's expansion is a corollary of Proposition 3. As a by-product, we obtain that it continues to hold whenever μ_q^p exists, in particular when A or B is positive semi-definite, a case that Smith's original derivation does not cover. Before we present his expression, we first introduce the normalized top-order invariant polynomials. For the rest of the paper, we shall adopt the following notation: $\boldsymbol{t} = (t_1, \ldots, t_r)$, $\boldsymbol{\kappa} = (k_1, \ldots, k_r)$, the k_i being nonnegative integers, $|\boldsymbol{\kappa}|$ will denote the sum of the parts of $\boldsymbol{\kappa}$, i.e., $|\boldsymbol{\kappa}| = \sum_{i=1}^r k_i$, and $\boldsymbol{t}^{\boldsymbol{\kappa}} = \prod_{i=1}^r t_i^{k_i}$.

Throughout the paper, we use the notation in Wilf [13] for coefficients in a generating function: the expression $[\boldsymbol{t}^{\boldsymbol{\kappa}}]D(\boldsymbol{t})$ denotes the coefficient of $\boldsymbol{t}^{\boldsymbol{\kappa}}$ in the power series expansion of the function

$$D(\boldsymbol{t}) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\kappa}|=k} d_{\boldsymbol{\kappa}} \boldsymbol{t}^{\boldsymbol{\kappa}}.$$

Using Wilf's notation, we can write $d_{\boldsymbol{\kappa}}$ as

$$d_{\boldsymbol{\kappa}} = [\boldsymbol{t}^{\boldsymbol{\kappa}}]D(\boldsymbol{t}).$$

For $n \times n$ symmetric matrices A_1 to A_r, we define the following generating function

$$D(\boldsymbol{t}) = |I_n - t_1 A_1 - \cdots - t_r A_r|^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\kappa}|=k} d_{\boldsymbol{\kappa}}(A_1, \cdots, A_r),$$

where $d_{\boldsymbol{\kappa}}(A_1, \ldots, A_r)$ is the normalized top-order invariant polynomial (see Chikuse [2] and Hillier, Kan, and Wang [4]). Assuming both A and B are positive definite and $\frac{n}{2} + p > q$, Smith [12] provides a triple infinite series expansion of μ_q^p in terms of the top-order invariant polynomials. In terms of our notation, his expression is

$$\mu_q^p = \frac{2^{p-q}\beta^q e^{-\frac{\mu'\mu}{2}}}{\alpha^p} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-p)_i (q)_j \Gamma\left(\frac{n}{2} + p - q + k\right)}{2^k \left(\frac{1}{2}\right)_k \Gamma\left(\frac{n}{2} + i + j + k\right)} d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu'), \tag{12}$$

where $\hat{A} = I_n - \alpha A$, $\hat{B} = I_n - \beta B$ with $0 < \alpha < 2/a_{\max}$, $0 < \beta < 2/b_{\max}$, $a_{\max}$ and $b_{\max}$ are the maximum eigenvalues of A and B, respectively, and $(a)_k = a(a+1)\cdots(a+k-1)$ is the usual Pochhammer symbol.

When p is an integer, (12) can be simplified to a double infinite series:

$$\mu_q^p = 2^{p-q}\beta^q e^{-\frac{\mu'\mu}{2}} p! \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q)_j \Gamma\left(\frac{n}{2} + p - q + k\right)}{2^k \left(\frac{1}{2}\right)_k \Gamma\left(\frac{n}{2} + p + j + k\right)} d_{p,j,k}(A, \hat{B}, \mu\mu'). \tag{13}$$

In addition, (13) also holds when A is a general symmetric matrix.
Since top-order invariant polynomials are in general difficult to evaluate, these infinite series expressions of μ_q^p are rarely used by researchers in practice. Recently, Hillier, Kan, and Wang [4] have made some progress in the

numerical computation of top-order invariant polynomials. For the special case when p is an integer and $\mu = 0_n$, (13) simplifies to a single infinite series, and Hillier, Kan, and Wang [4] provide an algorithm for numerical evaluation of μ_q^p. However, for the more general case, these infinite series expressions remain only of academic interest.

When proving (12), Smith relies on the assumption that both A and B are positive definite matrices and his proof does not go through when A or B is positive semi-definite. He attempts to deal with some special cases when B is positive semi-definite, but leaves the general case as an open question. The following proposition, which we obtain from Proposition 3 by a rearrangement, settles this question: as long as μ_q^p exists, Smith's formula continues to hold even when A or B is positive semi-definite. Therefore, with the moment existence conditions in Proposition 1 satisfied, we can safely use (12) and (13).

Proposition 5. *Suppose μ_q^p exists and $0 < \alpha \leq 1/a_{\max}$, $0 < \beta \leq 1/b_{\max}$. Then μ_q^p is given by (12) when p is not an integer, and by (13) when p is an integer (in which case A may be a general symmetric matrix and no restriction on α is needed).*

Our proof of Proposition 5 rests on the connection between the coefficients $h_{i,j}(\hat{A}, \hat{B})$ of Section 4 and the top-order invariant polynomials $d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu')$, together with the absolute convergence of the triple series established in the Appendix, which justifies the rearrangement. In view of the numerical difficulty of evaluating (12), Proposition 5 is mainly of theoretical interest; for computation, the double series of Proposition 3 with the algorithms of Section 4, or the integration method of Section 3, should be used.

6. Conclusion

In this paper, we present new results related to the mixed moments of ratios of quadratic forms in normal random variables. We first provide the necessary and sufficient conditions for the existence of mixed moments. We then present the integration and infinite series approaches for evaluating the mixed moments. In addition, we introduce a new and more efficient infinite series expansion of the mixed moments. For both the integration and infinite series approaches, we develop efficient numerical algorithms that are vastly superior than the existing ones in the literature.

Our results can be extended to deal with other problems. For example, it is rather straightforward to extend our results when the numerator is defined as $(x'Ax + b'x + c)^p$. For future research, it is of interest to generalize the integration and infinite series approaches to deal with ratios of quadratic forms in non-normal random variables.

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Appendix

Proof of Proposition 1: Since the proposition is known to be true for integer p, we only need to prove it for the case when p is not an integer, and we assume A is positive semi-definite in our proof. In addition, we only need to prove the necessary conditions for the existence of μ_q^p since the sufficient conditions in Proposition 1 have already been established by Roberts [9].

When B is positive definite, we let $\lambda_1 > 0$ be the largest eigenvalue of $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ and we have

$$\frac{x'Ax}{x'Bx} \leq \lambda_1 \Rightarrow \frac{x'Bx}{x'Ax} \geq \frac{1}{\lambda_1}.$$

Let $\lceil p \rceil$ be the smallest integer that is larger than p. Since

$$\frac{(x'Ax)^p}{(x'Bx)^q} = \left(\frac{x'Bx}{x'Ax}\right)^{\lceil p \rceil - p} \frac{(x'Ax)^{\lceil p \rceil}}{(x'Bx)^{q+\lceil p \rceil - p}} \geq \frac{1}{\lambda_1^{\lceil p \rceil - p}} \frac{(x'Ax)^{\lceil p \rceil}}{(x'Bx)^{q+\lceil p \rceil - p}},$$

we have

$$\mu_q^p \geq \frac{1}{\lambda_1^{\lceil p \rceil - p}} \mu_{q+\lceil p \rceil - p}^{\lceil p \rceil}.$$

As $\lceil p \rceil$ is an integer, $\mu_{q+\lceil p \rceil - p}^{\lceil p \rceil}$ exists if and only if $\frac{n}{2} + \lceil p \rceil > q + \lceil p \rceil - p$, or equivalently $\frac{n}{2} + p > q$. Therefore, when $\frac{n}{2} + p \leq q$, μ_q^p does not exist.

We now consider the case when B is positive semi-definite. For case (i) with $\tilde{A}_{12} = 0_{m\times(n-m)}$ and $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$, we have

$$\frac{(x'Ax)^p}{(x'Bx)^q} = \frac{(z'\tilde{A}z)^p}{(z_1'D_{1b}z_1)^q} = \frac{(z_1'\tilde{A}_{11}z_1)^p}{(z_1'D_{1b}z_1)^q}.$$

Since D_{1b} is a positive definite matrix, we can use the result from the positive definite B case to show that $\frac{m}{2} + p > q$ is both necessary and sufficient for μ_q^p to exist.

For case (iii) with $\tilde{A}_{22} \neq 0_{(n-m)\times(n-m)}$, we let $QD_aQ' = \tilde{A}$, where $D_a = \mathrm{Diag}(a_1, \ldots, a_r)$ with $a_1 \geq a_2 \geq \cdots \geq a_r > 0$ being the $r \leq n$ nonzero eigenvalues of $\tilde{A}$, and Q is an $n \times r$ matrix of the corresponding eigenvectors. Let $\tilde{z} = Q'z$; then we have

$$z'\tilde{A}z = \tilde{z}'D_a\tilde{z} = \sum_{i=1}^{r} a_i \tilde{z}_i^2.$$

Note that $\tilde{z}_i = q_i'z = q_{i,1}{}'z_1 + q_{i,2}{}'z_2$, where $q_i = [q_{i,1}',\ q_{i,2}']'$ is the i-th column of Q. Since $\tilde{A}_{22}$ is a nonzero matrix, at least one of the $q_{i,2}$'s is not a zero vector. Suppose $q_{j,2}$ is not a zero vector. Conditional on z_1, $\tilde{z}_j^2 \sim \sigma^2\chi_1^2(\lambda)$, where $\lambda = (q_{j,1}'z_1 + q_{j,2}{}'\mu_2)^2/\sigma^2$ with $\mu_2 = P_2'\mu$, and $\sigma^2 = q_{j,2}'q_{j,2}$.

For a noncentral chi-squared random variable with ν degrees of freedom and noncentrality parameter λ, the Poisson mixture representation gives, for any $p \geq 0$,

$$E[\chi_\nu^2(\lambda)^p] = e^{-\frac{\lambda}{2}} \sum_{k=0}^{\infty} \frac{(\lambda/2)^k}{k!} E[(\chi_{\nu+2k}^2)^p] \ \geq\ e^{-\frac{\lambda}{2}} E[(\chi_\nu^2)^p],$$

by keeping only the $k = 0$ term. Applying this bound conditionally on z_1 (with $\nu = 1$) and using the fact that $\tilde{z}'D_a\tilde{z} \geq a_j\tilde{z}_j^2$, we obtain

$$E\left[\frac{(\tilde{z}'D_a\tilde{z})^p}{(z_1'D_{1b}z_1)^q}\right] \geq a_j^p\, E\left[\frac{\tilde{z}_j^{2p}}{(z_1'D_{1b}z_1)^q}\right] \geq a_j^p\sigma^{2p}E[(\chi_1^2)^p]\, E\left[\frac{e^{-\frac{\lambda}{2}}}{(z_1'D_{1b}z_1)^q}\right].$$

On the event $\{z_1'z_1 \leq 1\}$ we have

$$\lambda = (q_{j,1}'z_1 + q_{j,2}'\mu_2)^2/\sigma^2 \leq 2\left(q_{j,1}'q_{j,1} + (q_{j,2}'\mu_2)^2\right)/\sigma^2 \equiv C,$$

so that

$$E\left[\frac{e^{-\frac{\lambda}{2}}}{(z_1'D_{1b}z_1)^q}\right] \geq e^{-\frac{C}{2}}\, E\left[\frac{1_{\{z_1'z_1 \leq 1\}}}{(z_1'D_{1b}z_1)^q}\right].$$

As $z_1'D_{1b}z_1 \leq b_1(z_1'z_1)$ and the density of $z_1 \sim N(P_1'\mu, I_m)$ is bounded away from zero on $\{z_1'z_1 \leq 1\}$, the last expectation is bounded below by a positive multiple of $\int_{\|z\|\leq 1}\|z\|^{-2q}\mathrm{d}z$, which is infinite when $2q\geq m$. It follows that μ_q^p does not exist when $m/2\leq q$. $\square$

Proof of Proposition 2: Note that

$$\begin{aligned}
&\frac{\partial^k\phi(t_1,t_2)}{\partial t_1^k}\\
=\;& \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(x'Ax)^k\exp(t_1x'Ax+t_2x'Bx)\exp\left(-\frac{(x-\mu)'(x-\mu)}{2}\right)(\mathrm{d}x)\\
=\;& \exp\left(-\frac{\mu'\mu}{2}\right)\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(x'Ax)^k\exp\left(-\frac{x'(I_n-2t_1A-2t_2B)x}{2}+x'\mu\right)(\mathrm{d}x).
\end{aligned}$$

Making a transformation of $w=L^{-1}x$, where L is an $n\times n$ matrix such that $(I_n-2t_1A-2t_2B)^{-1}=LL'$, we have $x'Ax=w'(L'AL)w=w'Rw$. Then denoting $\tilde{\mu}=L'\mu$, we obtain

$$\begin{aligned}
\frac{\partial^k\phi(t_1,t_2)}{\partial t_1^k} &= |L|\exp\left(\frac{\tilde{\mu}'\tilde{\mu}}{2}-\frac{\mu'\mu}{2}\right)\\
&\quad\times\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(w'Rw)^k\exp\left(-\frac{(w-\tilde{\mu})'(w-\tilde{\mu})}{2}\right)(\mathrm{d}w)\\
&= \phi(t_1,t_2)E[(w'Rw)^k], \qquad (14)
\end{aligned}$$

where we make use of the identities $x'\mu=w'\tilde{\mu}$ and $|L|=|I_n-2t_1A-2t_2B|^{-\frac{1}{2}}$, and $w\sim N(\tilde{\mu},I_n)$ in the second equality. Setting $t_1=0$ and $t_2=-t$ in (14) yields (3) and setting $t_1=-s$ and $t_2=-t$ yields (4). $\square$

Before proving Propositions 3 and 5, we collect two identities (Lemmas 1 and 2) and establish the absolute convergence result (Lemma 3) that will justify all interchanges of summation and integration below.

The first is a double-summation form of the Gauss hypergeometric theorem.

Lemma 1. *Suppose $d>a+b+c$. We have*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_i(b)_j(c)_{i+j}}{(d)_{i+j}i!j!}=\frac{\Gamma(d)\Gamma(d-a-b-c)}{\Gamma(d-a-b)\Gamma(d-c)}. \qquad (15)$$

Proof: Letting $k = i + j$, we have

$$\begin{aligned}
\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_i(b)_j(c)_{i+j}}{(d)_{i+j}i!j!} &= \sum_{k=0}^{\infty}\frac{(c)_k}{(d)_k k!}\sum_{i=0}^{k}\binom{k}{i}(a)_i(b)_{k-i} \\
&= \sum_{k=0}^{\infty}\frac{(c)_k(a+b)_k}{(d)_k k!} \\
&= \frac{\Gamma(d)\Gamma(d-a-b-c)}{\Gamma(d-c)\Gamma(d-a-b)},
\end{aligned}$$

where the second equality follows from the Chu-Vandermonde identity, and the last equality follows from the Gauss hypergeometric theorem when $d > a + b + c$. This completes the proof of Lemma 1. □

The second identity relates $d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu')$ to $d_{i,j,k}(\hat{A}_{11}, \hat{B}_{11}, \mu_1\mu_1')$ when A and B are block diagonal. Here and below, for an $n \times n$ matrix C, C_{11}, C_{12}, and C_{22} denote the blocks of C conformable with the partition $u = [u_1',\ u_2']'$ of u into its first m and last $n - m$ elements, μ_1 and μ_2 denote the first m and last $n - m$ elements of μ, and $\delta_1 = \mu_1'\mu_1$, $\delta_2 = \mu_2'\mu_2$, $\delta = \mu'\mu$.

Lemma 2. *Suppose* $A_{12} = 0_{m\times(n-m)}$, $A_{22} = 0_{(n-m)\times(n-m)}$, $B_{12} = 0_{m\times(n-m)}$, *and* $B_{22} = 0_{(n-m)\times(n-m)}$. *We have*

$$\begin{aligned}
d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu') &= \sum_{r=0}^{i}\sum_{s=0}^{j}\sum_{t=0}^{k}\frac{d_{r,s,t}(\hat{A}_{11}, \hat{B}_{11}, \mu_1\mu_1')}{(i-r)!(j-s)!(k-t)!} \\
&\quad \times \frac{\left(\frac{1}{2}\right)_k \delta_2^{k-t}}{\left(\frac{1}{2}\right)_t}\left(\frac{n-m}{2} + k - t\right)_{i-r+j-s}. \qquad (16)
\end{aligned}$$

Proof: In order to prove this identity, we first note that when $A_{12} = 0_{m\times(n-m)}$, $A_{22} = 0_{(n-m)\times(n-m)}$, $B_{12} = 0_{m\times(n-m)}$, and $B_{22} = 0_{(n-m)\times(n-m)}$, we have

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & I_{n-m} \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} \hat{B}_{11} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & I_{n-m} \end{bmatrix},$$

where $\hat{A}_{11} = I_m - \alpha A_{11}$ and $\hat{B}_{11} = I_m - \beta B_{11}$. Let $u = [u_1',\ u_2']' \sim N(0_n, I_n)$, where u_1 is the first m elements of u. We have $q_2 = u_2'u_2 \sim \chi^2_{n-m}$ and it is independent of u_1. Applying binomial expansions to $(u'\hat{A}u)^i = (u_1'\hat{A}_{11}u_1 +$

$q_2)^i$, $(u'\hat{B}u)^j = (u_1'\hat{B}_{11}u_1 + q_2)^j$, $(u'\mu\mu'u)^k = (\mu_1'u_1 + \mu_2'u_2)^{2k}$, and noting that $E[(u_1'\hat{A}_{11}u_1)^r(u_1'\hat{B}_{11}u_1)^s(u_1'\mu_1)^t] = 0$ when t is odd, we have

$$\begin{aligned} & E[(u'\hat{A}u)^i(u'\hat{B}u)^j(u'\mu\mu'u)^k] \\ = & \sum_{r=0}^{i}\sum_{s=0}^{j}\sum_{t=0}^{k}\binom{i}{r}\binom{j}{s}\binom{2k}{2t}E[(u_1'\hat{A}_{11}u_1)^r(u_1'\hat{B}_{11}u_1)^s(u_1'\mu_1)^{2t}] \\ & \times E[q_2^{i-r+j-s}(\mu_2'u_2)^{2k-2t}]. \end{aligned}$$

Using the fact that $(2k)! = 2^{2k}k!\left(\frac{1}{2}\right)_k$, we can write

$$\binom{2k}{2t} = \binom{k}{t}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_t\left(\frac{1}{2}\right)_{k-t}}.$$

For the second expectation, we note that

$$X = \frac{(\mu_2'u_2)^2}{(\mu_2'\mu_2)(u_2'u_2)} \sim \text{Beta}\left(\frac{1}{2}, \frac{n-m}{2}\right)$$

is independent of $q_2 = u_2'u_2$ and its $(k-t)$-th moment is given by

$$E[X^{k-t}] = \frac{\left(\frac{1}{2}\right)_{k-t}}{\left(\frac{n-m}{2}\right)_{k-t}}.$$

It follows that

$$\begin{aligned} E[q_2^{i-r+j-s}(\mu_2'u_2)^{2k-2t}] &= (\mu_2'\mu_2)^{k-t}E[q_2^{i-r+j-s+k-t}X^{k-t}] \\ &= \delta_2^{k-t}2^{i-r+j-s+k-t}\left(\frac{n-m}{2}\right)_{i-r+j-s+k-t}\frac{\left(\frac{1}{2}\right)_{k-t}}{\left(\frac{n-m}{2}\right)_{k-t}} \\ &= \delta_2^{k-t}2^{i-r+j-s+k-t}\left(\frac{1}{2}\right)_{k-t}\left(\frac{n-m}{2}+k-t\right)_{i-r+j-s}. \end{aligned}$$

With these results and the fact that

$$\begin{aligned} d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu') &= \frac{E[(u'\hat{A}u)^i(u'\hat{B}u)^j(u'\mu\mu'u)^k]}{2^{i+j+k}i!j!k!}, \\ d_{r,s,t}(\hat{A}_{11}, \hat{B}_{11}, \mu_1\mu_1') &= \frac{E[(u_1'\hat{A}_{11}u_1)^r(u_1'\hat{B}_{11}u_1)^s(u_1'\mu_1\mu_1'u_1)^t]}{2^{r+s+t}r!s!t!}, \end{aligned}$$

we can easily prove (16). This completes the proof of Lemma 2. $\square$

The following lemma is the central absolute convergence result. Recall from Eq. (48) of Hillier, Kan, and Wang [4] that for $n \times n$ symmetric matrices $A_1, \ldots, A_r$,

$$d_{k_1,\ldots,k_r}(A_1, \ldots, A_r) = \frac{E\left[\prod_{l=1}^{r}(u'A_l u)^{k_l}\right]}{2^{k_1+\cdots+k_r}\, k_1! \cdots k_r!}, \qquad u \sim N(0_n, I_n), \tag{17}$$

so that $d_{k_1,\ldots,k_r}(A_1, \ldots, A_r) \geq 0$ whenever $A_1, \ldots, A_r$ are all positive semi-definite. In particular, under the restrictions $0 < \alpha \leq 1/a_{\max}$ and $0 < \beta \leq 1/b_{\max}$ of Proposition 3 (with A positive semi-definite when p is not an integer), all the eigenvalues of $\hat{A}$ and $\hat{B}$ lie in $[0, 1]$, and $d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu') \geq 0$ for all i, j, k.

Lemma 3. *Let B be positive semi-definite with rank $m \geq 1$, $0 < \beta \leq 1/b_{\max}$, and let $c > 0$ be an arbitrary constant. Suppose that the existence condition of Proposition 1 is satisfied.*

(i) *If p is a positive integer and A is symmetric, then*

$$\begin{aligned} T_1 &\equiv \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(q)_j \Gamma\left(\frac{n}{2}+p-q+k\right) c^k}{2^k \left(\frac{1}{2}\right)_k \Gamma\left(\frac{n}{2}+p+j+k\right)}\, \bar{d}_{p,j,k} < \infty, \\ \bar{d}_{p,j,k} &\equiv \frac{E[\,|u'Au|^p (u'\hat{B}u)^j (u'\mu\mu'u)^k]}{2^{p+j+k}\, p!\, j!\, k!}. \end{aligned}$$

(ii) *If $p > 0$ is not an integer, A is positive semi-definite, and $0 < \alpha \leq 1/a_{\max}$, then, with F_i denoting either $|(-p)_i|$ or $C_p(1+i)^{\lceil p \rceil}|(-p)_i|$ for a constant C_p,*

$$T_2 \equiv \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{F_i (q)_j \Gamma\left(\frac{n}{2}+p-q+k\right) c^k}{2^k \left(\frac{1}{2}\right)_k \Gamma\left(\frac{n}{2}+i+j+k\right)}\, d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu') < \infty,$$

provided that, in the case of the weights $F_i = C_p(1+i)^{\lceil p \rceil}|(-p)_i|$, we additionally have $\mathrm{rank}(A) > 2\langle p \rangle$.

Note that $|d_{p,j,k}(A, \hat{B}, \mu\mu')| \leq \bar{d}_{p,j,k}$ by (17) and the triangle inequality, so part (i) bounds the series (13) and (11) termwise; similarly part (ii) bounds (12) and (10).

Proof: Without loss of generality (rotating u and μ as in the proof of Proposition 1), we may assume $B = \mathrm{Diag}(b_1, \ldots, b_m, 0'_{n-m})$ with $b_1 \geq \cdots \geq b_m > 0$, so that $\hat{B} = \mathrm{Diag}(\hat{B}_{11}, I_{n-m})$ with $\hat{b} \equiv \max_{i\le m}(1-\beta b_i) \in [0,1)$. Write $u = [u_1',\ u_2']'$, $q_1 = u_1'u_1 \sim \chi^2_m$, and $q_2 = u_2'u_2 \sim \chi^2_{n-m}$, independent of each other. Then

$$u'\hat{B}u \le \hat{b}q_1 + q_2, \qquad u'\mu\mu'u = (\mu'u)^2 \le \delta(q_1+q_2). \tag{18}$$

Throughout the proof we use the moment formula

$$E[q_1^x]E[q_2^y] = 2^{x+y}\left(\frac{m}{2}\right)_x \left(\frac{n-m}{2}\right)_y,$$

which is valid for all real $x > -\frac{m}{2}$ and $y > -\frac{n-m}{2}$, together with the Chu–Vandermonde identity $\sum_{t=0}^k \binom{k}{t}(a)_t(b)_{k-t} = (a+b)_k$.

(i) Partitioning A conformably and using $|u_1'A_{11}u_1| \le \|A_{11}\|q_1$, $|u_1'A_{12}u_2| \le \|A_{12}\|\sqrt{q_1q_2}$ (by the Cauchy–Schwarz inequality), and $|u_2'A_{22}u_2| \le \|A_{22}\|q_2$, where $\|\cdot\|$ denotes the spectral norm, we have $|u'Au| \le c_{11}q_1 + 2c_{12}\sqrt{q_1q_2} + c_{22}q_2$ with $c_{lm} = \|A_{lm}\|$, and hence, by the multinomial theorem,

$$|u'Au|^p \le \sum_{a+b+c=p} \frac{p!}{a!\,b!\,c!}\, c_{11}^a (2c_{12})^b c_{22}^c\; q_1^{a+\frac{b}{2}} q_2^{\frac{b}{2}+c},$$

the sum being over nonnegative integers a, b, c. Note that terms with $b>0$ occur only if $A_{12} \ne 0$, and terms with $c>0$ only if $A_{22} \ne 0_{(n-m)\times(n-m)}$. Combining this bound with (18), and expanding $(\hat{b}q_1+q_2)^j$ and $(q_1+q_2)^k$ binomially, the independence of q_1 and q_2 and the moment formula give

$$\begin{aligned}
\bar{d}_{p,j,k} \;&\le\; \frac{\delta^k}{p!\,j!\,k!} \sum_{a+b+c=p} \frac{p!\,c_{11}^a(2c_{12})^b c_{22}^c}{a!\,b!\,c!} \sum_{s=0}^{j}\sum_{t=0}^{k} \binom{j}{s}\binom{k}{t} \hat{b}^s \left(\frac{m}{2}\right)_{a+\frac{b}{2}+s+t} \\
&\quad\times \left(\frac{n-m}{2}\right)_{\frac{b}{2}+c+j-s+k-t} \\
&=\; \frac{\delta^k\left(\frac{n}{2}+p+j\right)_k}{p!\,j!\,k!} \sum_{a+b+c=p} w_{abc} \sum_{s=0}^{j}\binom{j}{s}\hat{b}^s(\nu_1)_s(\nu_2)_{j-s},
\end{aligned}$$

where $\nu_1 = \nu_1(a,b) = \frac{m}{2}+a+\frac{b}{2}$, $\nu_2 = \nu_2(b,c) = \frac{n-m}{2}+\frac{b}{2}+c$ (so that $\nu_1+\nu_2 = \frac{n}{2}+p$ for every (a,b,c)), $w_{abc} = \frac{p!\,c_{11}^a(2c_{12})^b c_{22}^c}{a!\,b!\,c!}\left(\frac{m}{2}\right)_{a+\frac{b}{2}}\left(\frac{n-m}{2}\right)_{\frac{b}{2}+c}$, and the equality follows from $\left(\frac{m}{2}\right)_{a+\frac{b}{2}+s+t} = \left(\frac{m}{2}\right)_{a+\frac{b}{2}}(\nu_1)_{s+t}$, the analogous factorization of the second Pochhammer symbol, and the Chu–Vandermonde identity applied to the sum over t. Substituting this bound into T_1 and using $\left(\frac{n}{2}+p+j\right)_k/\Gamma\left(\frac{n}{2}+p+j+k\right) = 1/\Gamma\left(\frac{n}{2}+p+j\right)$, the sum over k factors out as

$$\sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+p-q+k\right)(c\delta)^k}{2^k\left(\frac{1}{2}\right)_k k!} = \Gamma\left(\frac{n}{2}+p-q\right) {}_1F_1\left(\frac{n}{2}+p-q; \frac{1}{2}; \frac{c\delta}{2}\right) < \infty,$$

where the convergence holds for every value of $c\delta$, because the ratio of successive terms of the series is

$$\frac{\left(\frac{n}{2}+p-q+k\right)}{\left(\frac{1}{2}+k\right)(k+1)} \cdot \frac{c\delta}{2} \to 0$$

as $k \to \infty$ (equivalently, ${}_1F_1(a;b;z)$ is an entire function of z). For the sum over j, letting $r = j - s$ and exchanging the order of summation,

$$\begin{aligned}
& \sum_{j=0}^{\infty} \frac{(q)_j}{j!\,\Gamma\left(\frac{n}{2}+p+j\right)} \sum_{s=0}^{j} \binom{j}{s} \hat{b}^s (\nu_1)_s (\nu_2)_{j-s} \\
= \; & \sum_{s=0}^{\infty} \frac{(q)_s (\nu_1)_s \hat{b}^s}{\Gamma\left(\frac{n}{2}+p+s\right) s!} \sum_{r=0}^{\infty} \frac{(q+s)_r (\nu_2)_r}{\left(\frac{n}{2}+p+s\right)_r r!} \\
= \; & \sum_{s=0}^{\infty} \frac{(q)_s (\nu_1)_s \hat{b}^s}{\Gamma\left(\frac{n}{2}+p+s\right) s!} \frac{\Gamma\left(\frac{n}{2}+p+s\right)\Gamma(\nu_1-q)}{\Gamma\left(\frac{n}{2}+p-q\right)\Gamma(\nu_1+s)} \\
= \; & \frac{\Gamma(\nu_1-q)\,(1-\hat{b})^{-q}}{\Gamma(\nu_1)\Gamma\left(\frac{n}{2}+p-q\right)},
\end{aligned}$$

where the second equality follows from the Gauss hypergeometric theorem, using $\frac{n}{2}+p-\nu_2 = \nu_1$, and is applicable because $\nu_1 > q$; the last equality uses $(\nu_1)_s/\Gamma(\nu_1+s) = 1/\Gamma(\nu_1)$ and $\hat{b} < 1$. It remains to check that $\nu_1 = \frac{m}{2}+a+\frac{b}{2} > q$ for every (a,b,c) with $w_{abc} \neq 0$, and this is exactly what the existence condition of Proposition 1 delivers: if $\tilde{A}_{22} \neq 0_{(n-m)\times(n-m)}$, the smallest possible value of ν_1 is $\frac{m}{2}$ (at $a=b=0$) and the existence condition is $\frac{m}{2} > q$; if $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$ and $\tilde{A}_{12} \neq 0_{m\times(n-m)}$, then $c_{22}=0$ forces $c=0$, the smallest value of ν_1 is $\frac{m}{2}+\frac{p}{2}$ (at $a=0$, $b=p$), and the existence condition is $\frac{m+p}{2} > q$; if $\tilde{A}_{12} = 0_{m\times(n-m)}$ and $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$, then $b=c=0$, $\nu_1 = \frac{m}{2}+p$, and the existence condition is $\frac{m}{2}+p > q$; and if

$m = n$ (B positive definite), then q_2 is absent, $b = c = 0$, $\nu_1 = \frac{n}{2} + p$, and the existence condition is $\frac{n}{2} + p > q$. Summing over the finitely many (a, b, c) completes the proof of part (i).

(ii) Since A is positive semi-definite and $0 < \alpha \leq 1/a_{\max}$, all the eigenvalues of $\hat{A}$ lie in $[\hat{a}_0, 1]$ with $\hat{a}_0 = 1 - \alpha a_{\max} \in [0, 1)$, and $d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu') \geq 0$ by (17). We distinguish three cases according to the structure of B and $\tilde{A}$.

Case 1: B positive definite ($m = n$)*.* Here $\hat{b} = \max_i(1 - \beta b_i) = 1 - \beta b_{\min} < 1$, and we use the crude bounds $u'\hat{A}u \leq u'u$, $u'\hat{B}u \leq \hat{b}(u'u)$, and $(\mu'u)^2 \leq \delta(u'u)$ in (17) to obtain

$$d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu') \leq \frac{\hat{b}^j \delta^k E[(u'u)^{i+j+k}]}{2^{i+j+k} i! j! k!} = \frac{\hat{b}^j \delta^k \left(\frac{n}{2}\right)_{i+j+k}}{i! j! k!}.$$

Since $\left(\frac{n}{2}\right)_{i+j+k} \Big/ \Gamma\left(\frac{n}{2} + i + j + k\right) = 1/\Gamma\left(\frac{n}{2}\right)$, the triple sum T_2 factors:

$$T_2 \leq \frac{1}{\Gamma\left(\frac{n}{2}\right)} \left(\sum_{i=0}^{\infty} \frac{F_i}{i!}\right) \left(\sum_{j=0}^{\infty} \frac{(q)_j \hat{b}^j}{j!}\right) \left(\sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n}{2} + p - q + k\right) (c\delta)^k}{2^k \left(\frac{1}{2}\right)_k k!}\right),$$

and all three factors are finite: the second is $(1 - \hat{b})^{-q}$, the third converges for every value of $c\delta$ as in part (i), and for the first, $|(-p)_i|/i! = O(i^{-p-1})$ as $i \to \infty$, so that $\sum_i |(-p)_i|/i! < \infty$ for every $p > 0$. For the polynomial weights $F_i = C_p(1+i)^{\lceil p \rceil} |(-p)_i|$ the crude bound is not quite enough, and we use instead the refinement $u'\hat{A}u \leq \hat{a}_0' \, q_+ + q_0$, where q_0 is the squared norm of the projection of u onto the null space of A (of dimension $n_0 = n - \mathrm{rank}(A)$), $q_+ = u'u - q_0$, and $\hat{a}_0' = 1 - \alpha a_{\min}^+ < 1$ with $a_{\min}^+$ the smallest nonzero eigenvalue of A. Expanding $(\hat{a}_0' q_+ + q_0)^i$ binomially and proceeding as in part (i) yields

$$d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu') \leq \frac{\hat{b}^j \delta^k \left(\frac{n}{2} + i\right)_{j+k}}{j! k!} \sum_{c=0}^{i} \frac{\left(\frac{n_0}{2}\right)_c \left(\frac{n - n_0}{2}\right)_{i-c} (\hat{a}_0')^{i-c}}{c! \, (i - c)!},$$

and the sum over i of $F_i/\Gamma\left(\frac{n}{2} + i\right)$ times the last factor converges whenever $\mathrm{rank}(A) = n - n_0 > 2\langle p \rangle$, since the dominant terms (c close to i) contribute $O\left(i^{\lceil p \rceil} \cdot i^{-p-1} \cdot i^{-\mathrm{rank}(A)/2}\right) = O\left(i^{\langle p \rangle - 1 - \mathrm{rank}(A)/2}\right)$, which is summable, while the terms with $i - c$ large are geometrically small. (For the weights $F_i = |(-p)_i|$ no rank condition is needed.)

Case 2: B positive semi-definite with $m < n$ and $\tilde{A}_{22} \neq 0_{(n-m)\times(n-m)}$. The existence condition is $\frac{m}{2} > q$. Using $u'\hat{A}u \leq u'u = q_1 + q_2$ together with (18) and the same binomial/Chu–Vandermonde steps as in part (i),

$$d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu') \leq \frac{\delta^k \left(\frac{n}{2}+j\right)_{i+k}}{i!\,k!} \sum_{s=0}^{j} \frac{\left(\frac{m}{2}\right)_s \left(\frac{n-m}{2}\right)_{j-s} \hat{b}^s}{s!\,(j-s)!}.$$

Substituting into T_2 and using $\left(\frac{n}{2}+j\right)_{i+k} \Big/ \Gamma\left(\frac{n}{2}+i+j+k\right) = 1/\Gamma\left(\frac{n}{2}+j\right)$, the sums over i and k factor out exactly as before ($\sum_i F_i/i! < \infty$ and the entire ${}_1F_1$), leaving

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{(q)_j}{\Gamma\left(\frac{n}{2}+j\right)} \sum_{s=0}^{j} \frac{\left(\frac{m}{2}\right)_s \left(\frac{n-m}{2}\right)_{j-s} \hat{b}^s}{s!\,(j-s)!} \quad &= \quad \sum_{s=0}^{\infty} \frac{(q)_s \left(\frac{m}{2}\right)_s \hat{b}^s}{\Gamma\left(\frac{n}{2}+s\right) s!} \sum_{r=0}^{\infty} \frac{(q+s)_r \left(\frac{n-m}{2}\right)_r}{\left(\frac{n}{2}+s\right)_r r!} \\
&= \quad \frac{\Gamma\left(\frac{m}{2}-q\right)(1-\hat{b})^{-q}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}-q\right)} < \infty,
\end{aligned}$$

by the Gauss hypergeometric theorem (applicable because $\frac{m}{2} > q$), as in part (i). When the weights $F_i = C_p(1+i)^{\lceil p \rceil}|(-p)_i|$ are used, the sum over i is handled with the same null-space refinement as in Case 1, under the same condition $\mathrm{rank}(A) > 2\langle p \rangle$.

Case 3: B positive semi-definite with $m < n$ and $\tilde{A}_{12} = 0_{m\times(n-m)}$, $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$. The existence condition is $\frac{m}{2}+p > q$. In this case $\hat{A}$ and $\hat{B}$ are block diagonal with I_{n-m} lower blocks, and Lemma 2 expresses $d_{i,j,k}(\hat{A}, \hat{B}, \mu\mu')$ as a linear combination, with nonnegative coefficients, of the m-dimensional quantities $d_{r,s,t}(\hat{A}_{11}, \hat{B}_{11}, \mu_1\mu_1')$. Substituting (16) into T_2, exchanging the order of the (nonnegative) summations, and applying Lemma 1 exactly as in the display at the end of this appendix (with $|(-p)_r|$ or F_r in place of $(-p)_r$, all terms now being nonnegative), T_2 is bounded by a constant multiple of the corresponding m-dimensional sum with B_{11} positive definite, which is finite by Case 1 since $\frac{m}{2}+p > q$. (The Gauss-theorem step requires $\frac{m}{2}+p+t-q > 0$ for all $t \geq 0$, which holds.) This completes the proof of Lemma 3. □

Proof of Proposition 3: We need one more lemma, which evaluates the basic double integral that arises after expanding the integrand of (4) in a power series.

Lemma 4. *Suppose $\frac{n}{2}+p>q$ and i, j are nonnegative integers. We have*

$$\begin{aligned}&\frac{(-1)^{\lceil p\rceil}}{\Gamma(\langle p\rangle)\Gamma(q)}\int_0^\infty t^{q-1}(2t)^j\int_0^\infty s^{\langle p\rangle-1}\frac{\partial^{\lceil p\rceil}}{\partial s^{\lceil p\rceil}}(2s)^i(1+2s+2t)^{-\frac{n}{2}-i-j}\mathrm{d}s\mathrm{d}t\\ =\;&\frac{2^{p-q}(-p)_i(q)_j\Gamma\left(\frac{n}{2}+p-q\right)}{\Gamma\left(\frac{n}{2}+i+j\right)}.\end{aligned}$$

Proof: Using the Leibniz rule, we have

$$\begin{aligned}&\frac{\partial^{\lceil p\rceil}}{\partial s^{\lceil p\rceil}}(2s)^i(1+2s+2t)^{-\frac{n}{2}-i-j}\\ =\;&\sum_{l=0}^{\min[i,\lceil p\rceil]}\binom{\lceil p\rceil}{l}(-1)^l(-i)_l2^l(2s)^{i-l}\\ &\times(-1)^{\lceil p\rceil-l}\left(\frac{n}{2}+i+j\right)_{\lceil p\rceil-l}2^{\lceil p\rceil-l}(1+2s+2t)^{-\frac{n}{2}-i-j-\lceil p\rceil+l}\\ =\;&\frac{2^{\lceil p\rceil}(-1)^{\lceil p\rceil}}{\Gamma\left(\frac{n}{2}+i+j\right)}\sum_{l=0}^{\min[i,\lceil p\rceil]}\binom{\lceil p\rceil}{l}(-i)_l\Gamma\left(\frac{n}{2}+i+j+\lceil p\rceil-l\right)\\ &\times(2s)^{i-l}(1+2s+2t)^{-\frac{n}{2}-i-j-\lceil p\rceil+l}.\end{aligned}$$

When $i+\langle p\rangle>l$ (which is satisfied here because $l\le\min[i,\lceil p\rceil]$), we have

$$\begin{aligned}&\int_0^\infty s^{\langle p\rangle-1}(2s)^{i-l}(1+2s+2t)^{-\frac{n}{2}-i-j-\lceil p\rceil+l}\mathrm{d}s\\ =\;&\frac{2^{-\langle p\rangle}(1+2t)^{-\frac{n}{2}-p-j}\Gamma\left(\frac{n}{2}+p+j\right)\Gamma(i-l+\langle p\rangle)}{\Gamma\left(\frac{n}{2}+i+j+\lceil p\rceil-l\right)}.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_0^\infty s^{\langle p\rangle-1}\frac{\partial^{\lceil p\rceil}}{\partial s^{\lceil p\rceil}}(2s)^i(1+2s+2t)^{-\frac{n}{2}-i-j}\mathrm{d}s\\ =\;&\frac{2^p(-1)^{\lceil p\rceil}\Gamma(i-p)(1+2t)^{-\frac{n}{2}-p-j}\Gamma\left(\frac{n}{2}+p+j\right)}{\Gamma\left(\frac{n}{2}+i+j\right)}\sum_{l=0}^{\min[\lceil p\rceil,i]}\binom{\lceil p\rceil}{l}(-i)_l(i-p)_{\lceil p\rceil-l}\\ =\;&\frac{2^p(-1)^{\lceil p\rceil}\Gamma(i-p)(1+2t)^{-\frac{n}{2}-p-j}\Gamma\left(\frac{n}{2}+p+j\right)}{\Gamma\left(\frac{n}{2}+i+j\right)}\sum_{l=0}^{\lceil p\rceil}\binom{\lceil p\rceil}{l}(-i)_l(i-p)_{\lceil p\rceil-l}\\ =\;&\frac{2^p(-1)^{\lceil p\rceil}\Gamma(i-p)(-p)_{\lceil p\rceil}\Gamma\left(\frac{n}{2}+p+j\right)}{\Gamma\left(\frac{n}{2}+i+j\right)}(1+2t)^{-\frac{n}{2}-p-j},\end{aligned}$$

where the last equality follows from the Chu-Vandermonde identity. When $\frac{n}{2}+p>q$, we have

$$\int_0^\infty t^{q-1}(2t)^j(1+2t)^{-\frac{n}{2}-p-j}\mathrm{d}t=\frac{2^{-q}\Gamma\left(\frac{n}{2}+p-q\right)\Gamma(q+j)}{\Gamma\left(\frac{n}{2}+p+j\right)}.$$

Using this result, we obtain

$$\begin{aligned}
&\frac{(-1)^{\lceil p\rceil}}{\Gamma(\langle p\rangle)\Gamma(q)}\int_0^\infty t^{q-1}(2t)^j\int_0^\infty s^{\langle p\rangle-1}\frac{\partial^{\lceil p\rceil}}{\partial s^{\lceil p\rceil}}(2s)^i(1+2s+2t)^{-\frac{n}{2}-i-j}\mathrm{d}s\mathrm{d}t\\
=\;&\frac{(-1)^{\lceil p\rceil}}{\Gamma(\langle p\rangle)\Gamma(q)}\frac{2^p(-1)^{\lceil p\rceil}\Gamma(i-p)(-p)_{\lceil p\rceil}}{\Gamma\left(\frac{n}{2}+i+j\right)}2^{-q}\Gamma\left(\frac{n}{2}+p-q\right)\Gamma(q+j)\\
=\;&\frac{2^{p-q}(-p)_i(q)_j\Gamma\left(\frac{n}{2}+p-q\right)}{\Gamma\left(\frac{n}{2}+i+j\right)}.
\end{aligned}$$

This completes the proof of Lemma 4. □

For later use we record the identity

$$[t_1^it_2^j]\,|I_n-t_1A_1-t_2A_2|^{-\frac{1}{2}}\left(\mu'(I_n-t_1A_1-t_2A_2)^{-1}\mu\right)^N=\frac{N!\,d_{i,j,N}(A_1,A_2,\mu\mu')}{\left(\frac{1}{2}\right)_N},\tag{19}$$

which follows from

$$|I_n-t_1A_1-t_2A_2-t_3\mu\mu'|^{-\frac{1}{2}}=|I_n-t_1A_1-t_2A_2|^{-\frac{1}{2}}\left[1-t_3\,\mu'(I_n-t_1A_1-t_2A_2)^{-1}\mu\right]^{-\frac{1}{2}}$$

by expanding the last factor with the binomial series and matching the coefficients of t_3^N.

We first derive (11), so suppose that p is a positive integer. Starting from (3) applied to the pair $(A,\beta B)$, we have

$$\mu_q^p=\frac{\beta^q}{\Gamma(q)}\int_0^\infty t^{q-1}\,\frac{\partial^p}{\partial t_1^p}\phi(t_1,t_2)\bigg|_{t_1=0,\,t_2=-t}\mathrm{d}t,$$

where ϕ is the joint moment generating function of $x'Ax$ and $x'(\beta B)x$. Using the identity

$$I_n-2t_1A+2t\beta B=(1+2t)\left(I_n-uA-v\hat{B}\right),\qquad u=\frac{2t_1}{1+2t},\quad v=\frac{2t}{1+2t},$$

where $\hat{B} = I_n - \beta B$, together with $\frac{1}{1+2t} = 1 - v$, we obtain

$$
\begin{aligned}
& \phi(t_1, -t) \\
= \; & (1+2t)^{-\frac{n}{2}} |I_n - uA - v\hat{B}|^{-\frac{1}{2}} \exp\left(\frac{(1-v)\, \mu'(I_n - uA - v\hat{B})^{-1}\mu}{2} - \frac{\mu'\mu}{2} \right) \\
= \; & (1+2t)^{-\frac{n}{2}} \tilde{H}(u, v),
\end{aligned}
$$

with $\tilde{H}$ evaluated at $A_1 = A$, $A_2 = \hat{B}$. Since $u = 2t_1/(1+2t)$,

$$
\begin{aligned}
\left. \frac{\partial^p}{\partial t_1^p} \phi(t_1, -t) \right|_{t_1=0} &= (1+2t)^{-\frac{n}{2}} \left(\frac{2}{1+2t} \right)^p p! \, [u^p] \tilde{H}(u, v) \\
&= \frac{2^p p!}{(1+2t)^{\frac{n}{2}+p}} \sum_{j=0}^{\infty} \tilde{h}_{p,j}(A, \hat{B}) \left(\frac{2t}{1+2t} \right)^j ,
\end{aligned}
$$

where the termwise differentiation with respect to u at $u = 0$ is legitimate because, for each fixed $t \geq 0$, $\tilde{H}(u, v)$ is analytic in u in a neighborhood of the origin (the spectral radius of $uA + v\hat{B}$ is smaller than one for $|u|$ small, since $0 \leq v < 1$ and the eigenvalues of $\hat{B}$ lie in $[0, 1]$). Integrating term by term and using

$$
\int_0^{\infty} t^{q-1} (2t)^j (1+2t)^{-\frac{n}{2}-p-j} \mathrm{d}t = \frac{2^{-q} \Gamma\left(\frac{n}{2} + p - q\right) \Gamma(q+j)}{\Gamma\left(\frac{n}{2} + p + j\right)}
$$

(established in the proof of Lemma 4), we arrive at

$$
\mu_q^p = 2^{p-q} \beta^q p! \, \frac{\Gamma\left(\frac{n}{2} + p - q\right)}{\Gamma\left(\frac{n}{2} + p\right)} \sum_{j=0}^{\infty} \frac{(q)_j}{\left(\frac{n}{2} + p\right)_j} \tilde{h}_{p,j}(A, \hat{B}),
$$

which is (11). The termwise integration is justified by Tonelli's theorem: expanding

$$
\exp\left(\frac{(1-v)\mu'(I_n - uA - v\hat{B})^{-1}\mu}{2} \right) = \exp\left(\frac{\mu'(\cdot)^{-1}\mu}{2} \right) \exp\left(-\frac{v\, \mu'(\cdot)^{-1}\mu}{2} \right)
$$

as a product of two exponential series and extracting coefficients with the help of (19) below gives the exact representation

$$
\tilde{h}_{i,j}(A, \hat{B}) = e^{-\frac{\mu'\mu}{2}} \sum_{m=0}^{\infty} \sum_{w=0}^{j} \frac{(-1)^w (m+w)!}{2^{m+w} m! \, w! \left(\frac{1}{2}\right)_{m+w}} \, d_{i,j-w,m+w}(A, \hat{B}, \mu\mu'), \quad (20)
$$

so that $|\tilde{h}_{p,j}|$ is bounded by the same expression with $(-1)^w$ deleted and d replaced by $\bar{d}$; substituting this bound into the series of termwise absolute integrals, reindexing with $j = j' + w$, $k = m + w$, and using

$$\begin{aligned} (q)_{j'+w} &\leq (q)_{j'}(q+j')_w, \\ \left(\frac{n}{2}+p\right)_{j'+w} &\geq \left(\frac{n}{2}+p\right)_{j'}\left(\frac{n}{2}+p+j'\right)_w, \\ (q+j')_w &\leq \left(\frac{n}{2}+p+j'\right)_w \end{aligned}$$

(which holds because $q < \frac{n}{2} + p$), and $\sum_{w=0}^{k} \binom{k}{w} = 2^k$, the result is bounded by a constant multiple of T_1 of Lemma 3 with $c = 2$, which is finite. This proves (11).

We now derive (10), so suppose that p is not an integer and that A is positive semi-definite. Starting from (2) applied to the pair $(\alpha A, \beta B)$, we have

$$\mu_q^p = \frac{(-1)^{\lceil p \rceil}\beta^q}{\alpha^p \Gamma(\langle p \rangle)\Gamma(q)} \int_0^\infty \int_0^\infty s^{\langle p \rangle - 1} t^{q-1} \frac{\partial^{\lceil p \rceil}}{\partial s^{\lceil p \rceil}} \left[(1+2s+2t)^{-\frac{n}{2}} H(t_1,t_2)\right] \mathrm{d}s\mathrm{d}t,$$

where $t_1 = 2s/(1+2s+2t)$, $t_2 = 2t/(1+2s+2t)$, and H is evaluated at $A_1 = \hat{A}$, $A_2 = \hat{B}$. Indeed,

$$I_n + 2s\alpha A + 2t\beta B = (1+2s+2t)\left(I_n - t_1\hat{A} - t_2\hat{B}\right),$$

and

$$\frac{1}{1+2s+2t} = 1 - t_1 - t_2,$$

so that

$$e^{-\frac{\mu'\mu}{2}}\phi(-s,-t) = (1+2s+2t)^{-\frac{n}{2}} H(t_1,t_2)$$

exactly as in the integer case. Substituting the expansion

$$H(t_1,t_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} h_{i,j}(\hat{A},\hat{B}) t_1^i t_2^j$$

and noting that

$$(1+2s+2t)^{-\frac{n}{2}} t_1^i t_2^j = (2s)^i (2t)^j (1+2s+2t)^{-\frac{n}{2}-i-j},$$

termwise differentiation (legitimate by local uniform convergence: on the integration domain, $t_1, t_2 \geq 0$ and $t_1 + t_2 < 1$, so the spectral radius of

$t_1\hat{A} + t_2\hat{B}$ is less than one, with a uniform margin on compact subsets) followed by termwise integration and Lemma 4 gives

$$\mu_q^p = \frac{2^{p-q}\beta^q\Gamma\left(\frac{n}{2}+p-q\right)}{\alpha^p}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-p)_i(q)_j}{\Gamma\left(\frac{n}{2}+i+j\right)}h_{i,j}(\hat{A},\hat{B}),$$

which is (10). For the justification of the termwise integration, note first that in the Leibniz expansion of $\partial_s^{\lceil p\rceil}(2s)^i(1+2s+2t)^{-\frac{n}{2}-i-j}$ displayed in the proof of Lemma 4, each term has a finite integral against $s^{\langle p\rangle-1}t^{q-1}(2t)^j$, and the sum of the absolute values of these integrals differs from the absolute value of their (signed) sum, which is

$$2^{p-q}|(-p)_i|(q)_j\Gamma\left(\frac{n}{2}+p-q\right)\Gamma(q)\Gamma(\langle p\rangle)/\Gamma\left(\frac{n}{2}+i+j\right)$$

up to normalization, by at most a factor $C_p(1+i)^{\lceil p\rceil}$ (the Chu–Vandermonde sum $\sum_l\binom{\lceil p\rceil}{l}(-i)_l(i-p)_{\lceil p\rceil-l} = (-p)_{\lceil p\rceil}$ is replaced by the sum of the absolute values of its terms, and the ratio of the two grows at most like a polynomial of degree $\lceil p\rceil$ in i). Second, expanding H through the analogue of (20) (with shifts in both indices, since the exponent of H carries the factor $1-t_1-t_2$) bounds $|h_{i,j}|$ by a nonnegative combination of the $d_{i-u,j-v,m+u+v}(\hat{A},\hat{B},\mu\mu')$, and the reindexed series of termwise absolute integrals is bounded by a constant multiple of T_2 of Lemma 3 with weights $F_i = C_p(1+i)^{\lceil p\rceil}|(-p)_i|$ and $c=4$. By Lemma 3(ii), this is finite whenever $\operatorname{rank}(A) > 2\langle p\rangle$; in particular, whenever $\operatorname{rank}(A)\geq 2$, and, for $\operatorname{rank}(A)=1$, whenever $\langle p\rangle < \frac{1}{2}$. In the single remaining case ($\operatorname{rank}(A)=1$ and $\langle p\rangle\geq\frac{1}{2}$), (10) follows by analytic continuation in p: on each interval of non-integer values of p with $\frac{n}{2}+p>q$ fixed by the existence condition, both sides of (10) are analytic functions of p (the left side by differentiation under the expectation, the right side because the series converges locally uniformly in p by Lemma 3(ii) with weights $|(-p)_i|$), and they agree on the open subinterval where $\langle p\rangle<\frac{1}{2}$, hence everywhere on the interval. □

Proof of Proposition 5: We prove the proposition by showing that the right hand side of (12) is equal to the right hand side of (10); since the latter equals μ_q^p by Proposition 3, this establishes (12). All the series that appear below converge absolutely by Lemma 3 (every rearrangement below is dominated termwise by T_2 with weights $F_i = |(-p)_i|$, up to the harmless replacement of δ^k by $(c\delta)^k$ noted there), so the rearrangements and exchanges in the order of summation are justified. We first build a connection between $h_{i,j}(\hat{A},\hat{B})$ and $d_{i,j,k}(\hat{A},\hat{B},\mu\mu')$. Writing $H(t_1,t_2)$ as

$$
\begin{aligned}
& H(t_1,t_2) \\
= \; & e^{-\frac{\mu'\mu}{2}}|I_n - t_1\hat{A} - t_2\hat{B}|^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(1-t_1-t_2)^m}{2^m m!}(\mu'(I_n - t_1\hat{A} - t_2\hat{B})^{-1}\mu)^m \\
= \; & e^{-\frac{\mu'\mu}{2}}|I_n - t_1\hat{A} - t_2\hat{B}|^{-\frac{1}{2}} \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^l(t_1+t_2)^l}{2^{m+l} l! m!}(\mu'(I_n - t_1\hat{A} - t_2\hat{B})^{-1}\mu)^{m+l},
\end{aligned}
$$

we have

$$
\begin{aligned}
h_{i,j} &= [t_1^i t_2^j] H(t_1,t_2) \\
&= e^{-\frac{\mu'\mu}{2}} \sum_{m=0}^{\infty} \frac{1}{2^m m!}[t_1^i t_2^j] \sum_{l=0}^{i+j} \frac{(-1)^l(t_1+t_2)^l}{2^l l!} \\
&\quad \times |I_n - t_1\hat{A} - t_2\hat{B}|^{-\frac{1}{2}}(\mu'(I_n - t_1\hat{A} - t_2\hat{B})^{-1}\mu)^{m+l} \\
&= e^{-\frac{\mu'\mu}{2}} \sum_{m=0}^{\infty} \frac{1}{2^m m!} \sum_{u=0}^{i}\sum_{v=0}^{j} \frac{(-1)^{u+v}}{2^{u+v} u! v!} \\
&\quad \times [t_1^{i-u} t_2^{j-v}]|I_n - t_1\hat{A} - t_2\hat{B}|^{-\frac{1}{2}}(\mu'(I_n - t_1\hat{A} - t_2\hat{B})^{-1}\mu)^{m+u+v} \\
&= e^{-\frac{\mu'\mu}{2}} \sum_{m=0}^{\infty}\sum_{u=0}^{i}\sum_{v=0}^{j} \frac{(-1)^{u+v}}{2^{m+u+v} m! u! v!} \frac{(m+u+v)!}{\left(\frac{1}{2}\right)_{m+u+v}} d_{i-u,j-v,m+u+v}(\hat{A},\hat{B},\mu\mu'),
\end{aligned}
$$

where the third equality is obtained by writing $l = u+v$, and the last equality follows from (19). Using this expression of $h_{i,j}$, we can now pass from the right hand side of (12) to the right hand side of (10) as follows (the display is read from the first line to the last, each step being a rearrangement of an absolutely convergent series):

$$
\begin{aligned}
S &\equiv \frac{2^{p-q}\beta^q e^{-\frac{\mu'\mu}{2}}}{\alpha^p} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-p)_i (q)_j \Gamma\left(\frac{n}{2}+p-q+k\right)}{2^k \left(\frac{1}{2}\right)_k \Gamma\left(\frac{n}{2}+i+j+k\right)} d_{i,j,k}(\hat{A},\hat{B},\mu\mu') \\
&= \frac{2^{p-q}\beta^q e^{-\frac{\mu'\mu}{2}}}{\alpha^p} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-p)_i (q)_j \Gamma\left(\frac{n}{2}+p-q\right)}{2^k \left(\frac{1}{2}\right)_k \Gamma\left(\frac{n}{2}+i+j\right)} d_{i,j,k}(\hat{A},\hat{B},\mu\mu') \\
&\quad \times \sum_{u=0}^{k}\sum_{v=0}^{k-u} \frac{(-k)_{u+v}(-p+i)_u (q+j)_v}{\left(\frac{n}{2}+i+j\right)_{u+v} u! v!}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2^{p-q}\beta^q e^{-\frac{\mu'\mu}{2}}}{\alpha^p}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{u=0}^{k}\sum_{v=0}^{k-u}\frac{(-p)_{i+u}(q)_{j+v}\Gamma\left(\frac{n}{2}+p-q\right)}{2^k\left(\frac{1}{2}\right)_k\Gamma\left(\frac{n}{2}+i+u+j+v\right)}\\
&\quad\times\frac{k!(-1)^{u+v}}{(k-u-v)!u!v!}d_{i,j,k}(\hat{A},\hat{B},\mu\mu')\\
&= \frac{2^{p-q}\beta^q\Gamma\left(\frac{n}{2}+p-q\right)e^{-\frac{\mu'\mu}{2}}}{\alpha^p}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-p)_r(q)_s}{\Gamma\left(\frac{n}{2}+r+s\right)}\\
&\quad\times\sum_{m=0}^{\infty}\sum_{u=0}^{r}\sum_{v=0}^{s}\frac{(-1)^{u+v}(m+u+v)!}{2^{m+u+v}m!u!v!\left(\frac{1}{2}\right)_{m+u+v}}d_{r-u,s-v,m+u+v}(\hat{A},\hat{B},\mu\mu')\\
&= \frac{2^{p-q}\beta^q\Gamma\left(\frac{n}{2}+p-q\right)}{\alpha^p}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-p)_r(q)_s}{\Gamma\left(\frac{n}{2}+r+s\right)}h_{r,s}(\hat{A},\hat{B}),
\end{aligned}
$$

where the second last equality follows by setting $r=i+u$, $s=j+v$, $m=k-u-v$, and the first equality is obtained by using Lemma 1 with $a=-p+i$, $b=q+j$, $c=-k$, and $d=\frac{n}{2}+i+j$,

$$
\begin{aligned}
\sum_{u=0}^{k}\sum_{v=0}^{k-u}\frac{(-p+i)_u(q+j)_v(-k)_{u+v}}{\left(\frac{n}{2}+i+j\right)_{u+v}u!v!} &= \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{(-p+i)_u(q+j)_v(-k)_{u+v}}{\left(\frac{n}{2}+i+j\right)_{u+v}u!v!}\\
&= \frac{\left(\frac{n}{2}+p-q\right)_k}{\left(\frac{n}{2}+i+j\right)_k}.
\end{aligned}
$$

The last line is the right hand side of (10), which equals μ_q^p by Proposition 3; reading the display from the last line to the first therefore establishes (12). The derivation of (13) from (11) for integer p is completely analogous, using the connection (20) between $\tilde{h}_{p,j}$ and $d_{p,j,k}(A,\hat{B},\mu\mu')$ in place of the connection between $h_{i,j}$ and $d_{i,j,k}(\hat{A},\hat{B},\mu\mu')$, and Lemma 3(i) to justify the rearrangements; in this case A may be a general symmetric matrix and no restriction on α is needed. □

Proof of Proposition 4: We only prove the recurrence relation for $h_{i,j}$ because the proof of the recurrence relation for $\tilde{h}_{i,j}$ is almost identical. Let

$$
\begin{aligned}
P(t_1,t_2) &= 2t_1\frac{\partial\ln H(t_1,t_2)}{\partial t_1}+2t_2\frac{\partial\ln H(t_1,t_2)}{\partial t_2}\\
&= \mathrm{tr}(A(\boldsymbol{t})(I_n-A(\boldsymbol{t}))^{-1})+(1-t_1-t_2)\mu'(I_n-A(\boldsymbol{t}))^{-2}\mu\\
&\quad-\mu'(I_n-A(\boldsymbol{t}))^{-1}\mu
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \operatorname{tr}(A(\boldsymbol{t})^k) + (1 - t_1 - t_2) \sum_{k=0}^{\infty} (k+1)\mu' A(\boldsymbol{t})^k \mu - \sum_{k=0}^{\infty} \mu' A(\boldsymbol{t})^k \mu \\
&= \sum_{k=1}^{\infty} \operatorname{tr}(A(\boldsymbol{t})^k) + \sum_{k=1}^{\infty} k\mu' A(\boldsymbol{t})^k \mu - (t_1 + t_2) \sum_{k=0}^{\infty} (k+1)\mu' A(\boldsymbol{t})^k \mu;
\end{aligned}
$$

then it is easy to see that

$$
p_{i,j} = [t_1^i t_2^j] P(t_1, t_2) = \tau_{i,j} + (i+j)(\eta_{i,j} - \eta_{i-1,j} - \eta_{i,j-1}).
$$

Since

$$
P(t_1, t_2) H(t_1, t_2) = 2 \left[t_1 \frac{\partial H(t_1, t_2)}{\partial t_1} + t_2 \frac{\partial H(t_1, t_2)}{\partial t_2} \right],
$$

we can compare the coefficients of $t_1^i t_2^j$ on both sides to obtain the recurrence relation:

$$
\sum_{\nu_1=0}^{i} \sum_{\substack{\nu_2=0 \\ \nu_1+\nu_2>0}}^{j} p_{\nu_1,\nu_2} h_{i-\nu_1, j-\nu_2} = 2(i+j) h_{i,j}.
$$

$\square$

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