Errata and Clarifications for "On the expectations of equivariant matrix-valued functions of Wishart and inverse Wishart matrices"

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This note lists corrections to the published version of the paper. Items in Section 1 are genuine errors (typographical or otherwise); items in Section 2 are clarifications of statements that are correct as printed but whose justification was omitted or is easily misread. Equation numbers refer to the published version. All corrections have been checked against an independent implementation of the recurrences (exact symbolic computation) and, where relevant, against Monte Carlo simulation. Red text is the printed version, green text is the corrected version.

1 Corrections

Mathematical content

1. **Equations (45) and (51) (Proposition 4 and Theorem 2), pages 11–12.** The upper limit of the product is wrong; there is one factor too many. The correct statements are

$$\mathcal{C}_{k,s} = \left(\prod_{j=1}^{k-s-1} D_{k-j,a} \right) D_{s,b} \mathcal{H}_s, \qquad s = 1, \dots, k-1, \tag{45}$$

$$\mathcal{C}_{k,s} = (2s)^{-1} \left(\prod_{j=1}^{k-s-1} D_{k-j,a} \right) D_{s,b} D_{s,21} \mathcal{C}_s D_{s,12}, \qquad s = 1, \dots, k-1, \tag{51}$$

where the product is taken in the order $D_{k-1,a} D_{k-2,a} \cdots D_{s+1,a}$ and an empty product (the case $s = k-1$) is the identity. With the printed upper limit $k-s$ the matrices are not even conformable (e.g. for $k = 2$, $s = 1$ the product would be $D_{1,a} D_{1,b}$, a 2×1 times 2×1 product). Equations (43), (44), (46), (52)–(54), and the worked examples (56)–(61) are correct as printed and are consistent with the corrected (45) and (51).

2. **Equation (A24), page 25.** Both occurrences of $p_\lambda(W)$ should be $p_\lambda(W^{-1})$:

$$\begin{aligned} DW^{-r} p_\lambda(W^{-1}) = & -\frac{1}{2} \sum_{j=1}^{r} W^{-r-1+j} p_j(W^{-1}) p_\lambda(W^{-1}) - \binom{r}{2} W^{-r-1} p_\lambda(W^{-1}) \\ & - \sum_{i=1}^{\ell(\lambda)} \lambda_i W^{-r-1-\lambda_i} p_{\lambda_{(i)}}(W^{-1}). \end{aligned} \tag{A24}$$

3. **Equation (A30), page 25.** The exponent in the first sum is wrong:

$$\tilde{D}W^{-r}p_\lambda(W^{-1}) = -\sum_{j=1}^{r} W^{-r-1+j} p_j(W^{-1})p_\lambda(W^{-1}) - \sum_{i=1}^{\ell(\lambda)} \lambda_i W^{-r-1-\lambda_i} p_{\lambda_{(i)}}(W^{-1}). \quad \text{(A30)}$$

(The printed version has W^{-r-1-j}.) This is consistent with (A22) and with the recurrence (87), which are correct as printed.

4. **Equation (91), page 19.** The sub-matrices in the update formula for the complex case should be those of D_{-k}, not of $\tilde{D}_k$ (exactly as in (77) for the real case):

$$\tilde{C}_{k+1} = \left[\tilde{D}_{k,a}\tilde{C}_k,\ k^{-1}\tilde{D}_{k,b}D_{-k,21}\tilde{C}_k D_{-k,12}\right]. \quad (91)$$

5. **Page 7, line 4.** There is a spurious bracket. $\mathbb{E}[2W^2\operatorname{tr}(W)] + W\operatorname{tr}(W^2)]$ should read $\mathbb{E}[2W^2\operatorname{tr}(W) + W\operatorname{tr}(W^2)]$ (this is $\mathbb{E}[L_{(2,1)}(W)]$ in the notation of (25)).

6. **Equation (38), page 10.** There is a spurious period at the end of the first line of the display (after "$p_\lambda(W)]$"). The equation is otherwise correct.

Text and references

7. Page 2, line 1: "throughout most this paper" → "throughout most of this paper".

8. Page 13, line after (59): "similar computations using D_3 and $\mathcal{C}_3$ produces" → "…produce".

9. Page 17, line before Corollary 2: "Similar to Corollary 1, we have." → "Similar to Corollary 1, we have:".

10. Page 19 (Section 7, line 5) and reference list, page 20: "Haagerup and Thosbjørnsen (2003)" → "Haagerup and Thorbjørnsen (2003)". The reference should read: Haagerup, U., & Thorbjørnsen, S. (2003). Random matrices with complex Gaussian entries. *Expositiones Mathematicae*, 21, 293–337.

11. Page 7 (Section 3, second paragraph) and reference list, page 21: "Lu and Richard (2001)" → "Lu and Richards (2001)", and in the reference "MacMahaon's master theorem" → "MacMahon's master theorem". The reference should read: Lu, I.-L., & Richards, D. St. P. (2001). MacMahon's master theorem, representation theory, and moments of Wishart distributions. *Advances in Applied Mathematics*, 27, 531–547.

12. Reference list, page 21, Haff (1981): "…with application with applications in regression" → "…with applications in regression".

2 Clarifications (no change to results)

1. **Equation (82), Remark 9, page 18.** Equation (82) is correct, but it does *not* follow from (77) by simply inverting: (77) gives $\tilde{C}_{k+1} = \tilde{D}_k \operatorname{diag}(\tilde{C}_k, \tilde{\mathcal{H}}_k)$, hence $\tilde{C}_{k+1}^{-1} = \operatorname{diag}(\tilde{C}_k^{-1}, \tilde{\mathcal{H}}_k^{-1})D_{-k}$, whereas (82) states $\tilde{C}_{k+1}^{-1} = D_{-k}\operatorname{diag}(\tilde{C}_k^{-1}, \tilde{\mathcal{H}}_k^{-1})$. The two agree because D_{-k} commutes with $\operatorname{diag}(\tilde{C}_k, \tilde{\mathcal{H}}_k)$; equivalently, in the forward case, D_k commutes with $\operatorname{diag}(\mathcal{C}_k, \mathcal{H}_k)$, so that

$$\mathcal{C}_{k+1} = D_k \operatorname{diag}(\mathcal{C}_k, \mathcal{H}_k) = \operatorname{diag}(\mathcal{C}_k, \mathcal{H}_k)\, D_k.$$

The reason is the following intertwining property. Write the recurrence (28) as $\mathbb{E}[WF(W)] = \Sigma\,\mathbb{E}[\mathcal{D}F(W)]$, where $\mathcal{D}$ is the first-order differential operator $\mathcal{D}F = nF + 2\sum_{s\le t} e_{st}W\,\partial_{e_{st}}F$ (with e_{st} the unit symmetric matrices), so that D_k is the matrix of $\mathcal{D}$ on the span of the entries of $[\mathcal{L}_k(W);\, p^{(k)}(W)\otimes I_m]$. Differentiating $\mathbb{E}[F(W)]$ with respect to Σ (using $\partial \log f(W;\Sigma)/\partial\Sigma$) gives the identity $\mathbb{E}[WF(W)] = \Sigma\,\mathcal{D}_\Sigma\mathbb{E}[F(W)]$, with the same operator $\mathcal{D}$ now acting on functions of Σ. Hence $\mathbb{E}_n[\mathcal{D}F] = \mathcal{D}_\Sigma\mathbb{E}_n[F]$, and in coefficient form this is precisely the commutation of D_k with $\operatorname{diag}(\mathcal{C}_k, \mathcal{H}_k)$ (and of D_{-k} with $\operatorname{diag}(\tilde{\mathcal{C}}_k, \tilde{\mathcal{H}}_k)$). Full details are given in the companion note on the β-Wishart extension (Hillier and Kan, 2026, Theorem 5.2 and Corollaries 5.3 and 5.10; the commutation is in fact an algebraic identity valid for every $\alpha > 0$, Theorem 7.7 there). The Matlab and R programs use (82) and are therefore correct.

2. **Remark 1, page 3 (existence of inverse moments).** The condition $n - m + 1 > 2k$, equivalently $\tilde{n} > 2(k-1)$, is correct for the real case. For the complex case (Section 7) the corresponding condition is $\hat{n} > k - 1$, i.e. $n - m + 1 > k$. In the general β-Wishart case, with $\alpha = 2/\beta$ and $\tilde{n} = n - m + 1 - \alpha$, the condition is $\tilde{n} > (k-1)\alpha$, equivalently $(n-m+1)/\alpha > k$. We mention this because the condition "$\tilde{n} > 2k$" used in versions 1.1–2.3 of our Matlab programs and in version 1.1 of the R package `wishmom` is too strict (by one unit of α) and returns `NaN` for some moments that do exist; this is corrected in version 3.0 of the programs.

3. **Page 16, line after (68).** The sentence "Provided—as we assume—$\tilde{n} > 2k$, D_{-k} is a diagonally dominant matrix" gives a sufficient condition for the nonsingularity of D_{-k}; it is in fact also necessary in the sense that $\det D_{-k}$ vanishes at $\tilde{n} = 2k$. The distinct eigenvalues of D_{-k} are $\tilde{n} - c$, where c ranges over the contents $2j - i$ of the cells (i,j) (in zero-based coordinates) that can be added to some partition of k, i.e. over $\{2j - i : (i,j) \in S_k\}$, where

$$S_k = \big\{(i,j) \in \mathbb{Z}^2_{\ge 0} :\ (i+1)(j+1) \le k+1\big\} \setminus \{(0,0)\};$$

the largest such content is $2k$. The cell $(0,0)$ must be excluded, because it is addable only to the empty diagram: already for $k = 1$, $D_1 - nI = [\begin{smallmatrix}1 & 1\\ 2 & 0\end{smallmatrix}]$ has eigenvalues 2 and -1, not $2, 0, -1$. (The value 0 can still occur in $\{2j - i : (i,j) \in S_k\}$ for $k \ge 5$, through the cell $(2,1)$; what is excluded is the cell $(0,0)$, not the number 0.) In fact the characteristic polynomial of $D_k - nI$ is

$$\det\big(tI - (D_k - nI)\big) = \prod_{\lambda\vdash k}\ \prod_{(i,j)\in\mathcal{A}(\lambda)} (t - 2j + i),$$

where $\mathcal{A}(\lambda)$ is the set of addable cells of λ. This is the fact exploited by `dkimapr.m` in the programs, through `unid.m`, which does build the list S_k correctly.

Verification

Every displayed matrix in the paper ((35)–(37), (56), (58), (60)–(61), (66)–(68), (70)–(71), (78)–(81)) and the unbiased estimators (63)–(64), (84)–(85) were recomputed with an independent symbolic implementation of Theorem 1 and agree with the printed values. The corrected equations (45), (51), (A24), (A30) and (91) were checked in the same way.