Errata for "On the Moments of Ratios of Quadratic Forms in Normal Random Variables"

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This note lists corrections to our paper. None of the corrections affects the validity of the propositions or of the final formulae used for computation, with the exception of Eq. (13), which as printed produces incorrect numerical values and should be replaced by the corrected version given in item 2 below. Page, equation, and reference numbers refer to the published version of the paper.

1. **Page 230, Proposition 1, part (2)(ii).** A tilde is missing on A_{22}. The condition "$\tilde{A}_{12} \neq 0_{m\times(n-m)}$ and $A_{22} = 0_{(n-m)\times(n-m)}$" should read "$\tilde{A}_{12} \neq 0_{m\times(n-m)}$ and $\tilde{A}_{22} = 0_{(n-m)\times(n-m)}$."

2. **Page 234, Proposition 4, Eq. (13).** The term $\Gamma\left(\frac{n}{2}+j\right)$ in the denominator of the summand is incorrect and should be the Pochhammer symbol $\left(\frac{n}{2}+p\right)_j$. The equation currently reads

$$\mu_q^p = \frac{2^{p-q}\beta^q p!\Gamma\left(\frac{n}{2}+p-q\right)}{\Gamma\left(\frac{n}{2}+p\right)} \sum_{j=0}^{\infty} \frac{(q)_j}{\Gamma\left(\frac{n}{2}+j\right)} \tilde{h}_{p,j}(A,\hat{B}),$$

and it should read

$$\mu_q^p = \frac{2^{p-q}\beta^q p!\Gamma\left(\frac{n}{2}+p-q\right)}{\Gamma\left(\frac{n}{2}+p\right)} \sum_{j=0}^{\infty} \frac{(q)_j}{\left(\frac{n}{2}+p\right)_j} \tilde{h}_{p,j}(A,\hat{B}).$$

3. **Page 235, paragraph following the short recursions for $G_{i,k-i}$ and $g_{i,k-i}$.** In the last sentence of the paragraph, "particularly desirable for the computation of μ_p^q" should read "particularly desirable for the computation of μ_q^p."

4. **Page 236, proof of Proposition 1.** In the sentence "As $\lceil p \rceil$ is an integer, $\mu_{q+\lceil p\rceil-p}^{\lceil p\rceil}$ exists if and only $\frac{n}{2}+\lceil p\rceil > q+\lceil p\rceil - p, \ldots$" the word "if" is missing after "only."

5. **Page 236, proof of Proposition 1, case (iii).** In the p-th moment formula for the noncentral chi-squared random variable, the second argument of the confluent

hypergeometric function should be $\frac{\nu}{2}$ rather than $\frac{1}{2}$. The display currently reads

$$E[\chi^2_\nu(\lambda)^p] = \frac{2^p e^{-\frac{\lambda}{2}} \Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)} {}_1F_1\left(\frac{2p+\nu}{2}; \frac{1}{2}; \frac{\lambda}{2}\right),$$

and it should read

$$E[\chi^2_\nu(\lambda)^p] = \frac{2^p e^{-\frac{\lambda}{2}} \Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)} {}_1F_1\left(\frac{\nu}{2}+p; \frac{\nu}{2}; \frac{\lambda}{2}\right).$$

The formula as printed is valid only for $\nu = 1$, which is the only case used in the remainder of the proof, so the subsequent arguments are unaffected. In addition, in the expression $\lambda = (q'_{j,1}z_1 + q'_{j,2}\mu_2)^2/\sigma^2$, the vector μ_2 denotes the mean of z_2, i.e., $\mu_2 = P'_2\mu$; this notation is used without having been defined.

6. **Page 237, proof of Proposition 2.** There are several corrections in this proof.

   (a) On the left hand side of the first display, $\frac{\partial\phi(t_1,t_2)}{\partial t_1^k}$ should read $\frac{\partial^k\phi(t_1,t_2)}{\partial t_1^k}$.

   (b) The term $\exp(x'\mu)$ was erroneously dropped from the second line of the first display, and the accompanying justification "where the term $\exp(x'\mu)$ in the integrand vanishes because of symmetry" is incorrect (the term does not vanish). The second line of the first display should read

   $$\exp\left(-\frac{\mu'\mu}{2}\right)\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(x'Ax)^k\exp\left(-\frac{x'(I_n-2t_1A-2t_2B)x}{2}+x'\mu\right)(\mathrm{d}x).$$

   (c) The condition "L is an $n\times n$ matrix such that $(I_n - 2t_1A - 2t + 2B)^{-1} = LL'$" contains a typographical error and should read "$(I_n - 2t_1A - 2t_2B)^{-1} = LL'$."

   (d) Consistent with (b), the sentence following Eq. (14), "where we introduce an extra term of $\exp(-w'\tilde{\mu})$ in the first equality by using the symmetry argument, and we make use of the identity $|L| = |I_n - 2t_1A - 2t_2B|^{-\frac{1}{2}}$ in the second equality," should read "where we make use of the identities $x'\mu = w'\tilde{\mu}$ and $|L| = |I_n - 2t_1A - 2t_2B|^{-\frac{1}{2}}$, and $w \sim N(\tilde{\mu}, I_n)$ in the second equality." No symmetry argument is involved: under the transformation $x = Lw$ we simply have $x'\mu = w'L'\mu = w'\tilde{\mu}$.

7. **Page 239, proof of Proposition 3.** In the sentence "With these inequalities, we derive an upper bound on $|d_{p,j,k}(\hat{A},\hat{B},\mu\mu')|$ for this case," the subscripts should be i, j, k, i.e., $|d_{i,j,k}(\hat{A},\hat{B},\mu\mu')|$.

8. **Page 239, bound on $|d_{i,j,k}(\hat{A},\hat{B},\mu\mu')|$.** In the fifth line of the display, the binomial coefficient $\binom{j+k}{t}$ is missing from the inner summation. The line currently reads

$$\frac{\hat{b}^j\delta^k}{i!j!k!}\sum_{s=0}^{i}\binom{i}{s}\left(\frac{1}{2}\right)_s\left(\frac{n-1}{2}\right)_{i-s}\hat{a}^s\sum_{t=0}^{j+k}\left(\frac{1}{2}+s\right)_t\left(\frac{n-1}{2}+i-s\right)_{j+k-t},$$

and it should read

$$\frac{\hat{b}^j\delta^k}{i!j!k!}\sum_{s=0}^{i}\binom{i}{s}\left(\frac{1}{2}\right)_s\left(\frac{n-1}{2}\right)_{i-s}\hat{a}^s\sum_{t=0}^{j+k}\binom{j+k}{t}\left(\frac{1}{2}+s\right)_t\left(\frac{n-1}{2}+i-s\right)_{j+k-t}.$$

The last line of the display, which follows from the Chu–Vandermonde identity applied to this sum, is correct as printed.

9. **Pages 240–241, bound on $|d_{i,j,k}(\hat{A},\hat{B},\mu\mu')|$ for the case $A_{22}\neq 0_{(n-m)\times(n-m)}$.** In the second line of the display, the factor 2^{i+j+k} should divide rather than multiply, because the expectations $E[q_1^{r+s}]$ and $E[q_2^{i+k-r+j-s}]$ on that line have not yet been evaluated. The line currently reads

$$\frac{2^{i+j+k}\delta^k}{i!j!k!}\sum_{r=0}^{i+k}\sum_{s=0}^{j}\binom{i+k}{r}\binom{j}{s}\hat{b}^sE[q_1^{r+s}]E[q_2^{i+k-r+j-s}],$$

and it should read

$$\frac{\delta^k}{2^{i+j+k}i!j!k!}\sum_{r=0}^{i+k}\sum_{s=0}^{j}\binom{i+k}{r}\binom{j}{s}\hat{b}^sE[q_1^{r+s}]E[q_2^{i+k-r+j-s}].$$

The last line of the display is correct as printed.

10. **Page 241, bound on $|g_i|$.** In the fourth and fifth lines of the display, the Pochhammer symbol $\left(\frac{m}{2}\right)_s$ should appear in the numerator rather than in the denominator, i.e., the factor

$$\frac{(q)_s\hat{b}^s}{\Gamma\left(\frac{n}{2}+s\right)\left(\frac{m}{2}\right)_s s!}\qquad\text{should read}\qquad\frac{(q)_s\left(\frac{m}{2}\right)_s\hat{b}^s}{\Gamma\left(\frac{n}{2}+s\right)s!}$$

in both lines. The final expression for d is correct as printed.

11. **Page 241, last display.** On the right hand side of the identity, the summand indices do not match the summation indices r, s, t: the factors $(-p)_i(q)_j$ and $d_{i,j,k}(\hat{A}_{11},\hat{B}_{11},\mu_1\mu_1')$ should read $(-p)_r(q)_s$ and $d_{r,s,t}(\hat{A}_{11},\hat{B}_{11},\mu_1\mu_1')$. The display should read

$$\begin{aligned}&\frac{2^{p-q}\beta^q e^{-\frac{\delta}{2}}}{\alpha^p}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-p)_i(q)_j\Gamma\left(\frac{n}{2}+p-q+k\right)}{2^k\left(\frac{1}{2}\right)_k\Gamma\left(\frac{n}{2}+i+j+k\right)}d_{i,j,k}(\hat{A},\hat{B},\mu\mu')\\ =\ &\frac{2^{p-q}\beta^q e^{-\frac{\delta_1}{2}}}{\alpha^p}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-p)_r(q)_s\Gamma\left(\frac{m}{2}+p-q+t\right)}{2^t\left(\frac{1}{2}\right)_t\Gamma\left(\frac{m}{2}+r+s+t\right)}d_{r,s,t}(\hat{A}_{11},\hat{B}_{11},\mu_1\mu_1').\end{aligned}$$

12. **Page 243, proof of Lemma 3.** In the display following "With these results and the fact that," a subscript is missing in the first expectation of the second equation: $E[(u_1'\hat{A}_{11}u)^r(u_1'\hat{B}_{11}u_1)^s(u_1'\mu_1\mu_1'u_1)^t]$ should read $E[(u_1'\hat{A}_{11}u_1)^r(u_1'\hat{B}_{11}u_1)^s(u_1'\mu_1\mu_1'u_1)^t]$.

13. **Page 243, display following "Substituting (20) into (10), we obtain."** In the last factor of the display, the Pochhammer subscript $i-r+j-v$ should read $i-r+j-s$ (the index v is introduced only in the subsequent change of variables), i.e., $\left(\frac{n-m}{2}+k-t\right)_{i-r+j-v}$ should read $\left(\frac{n-m}{2}+k-t\right)_{i-r+j-s}$.

14. **Page 243, proof of Proposition 4, first display.** For consistency of notation, $(\mu'(I_n - t_1A_1 - t_2A_2)^{-1}\mu)^m$ in the first line should read $(\mu'(I_n - t_1\hat{A} - t_2\hat{B})^{-1}\mu)^m$, since the generating function H is evaluated here at $A_1 = \hat{A}$ and $A_2 = \hat{B}$.

15. **Page 244, proof of Proposition 4, second display.** In the third equality of the long display, the factors $(-p+i)_u(q+j)_v$ are superfluous, as they have already been absorbed into $(-p)_{i+u}(q)_{j+v}$ (note that $(-p)_{i+u} = (-p)_i(-p+i)_u$ and $(q)_{j+v} = (q)_j(q+j)_v$). The summand of the third equality currently reads

$$\frac{(-p)_{i+u}(q)_{j+v}\Gamma\left(\frac{n}{2}+p-q\right)}{2^k\left(\frac{1}{2}\right)_k\Gamma\left(\frac{n}{2}+i+u+j+v\right)} \times \frac{k!(-1)^{u+v}(-p+i)_u(q+j)_v}{(k-u-v)!u!v!}d_{i,j,k}(\hat{A},\hat{B},\mu\mu'),$$

and it should read

$$\frac{(-p)_{i+u}(q)_{j+v}\Gamma\left(\frac{n}{2}+p-q\right)}{2^k\left(\frac{1}{2}\right)_k\Gamma\left(\frac{n}{2}+i+u+j+v\right)} \times \frac{k!(-1)^{u+v}}{(k-u-v)!u!v!}d_{i,j,k}(\hat{A},\hat{B},\mu\mu').$$

16. **Page 244, proof of Proposition 5, first display.** In the third line of the expansion of $P(t_1,t_2)$, the last summation should start at $k=0$ rather than $k=1$, because $\mu'(I_n - A(\boldsymbol{t}))^{-1}\mu = \sum_{k=0}^{\infty}\mu' A(\boldsymbol{t})^k\mu$, where $\boldsymbol{t} = (t_1,t_2)$ and $A(\boldsymbol{t}) = t_1A_1 + t_2A_2$. The third line currently reads

$$\sum_{k=1}^{\infty}\operatorname{tr}(A(\boldsymbol{t})^k) + (1-t_1-t_2)\sum_{k=0}^{\infty}(k+1)\mu' A(\boldsymbol{t})^k\mu - \sum_{k=1}^{\infty}\mu' A(\boldsymbol{t})^k\mu,$$

and it should read

$$\sum_{k=1}^{\infty}\operatorname{tr}(A(\boldsymbol{t})^k) + (1-t_1-t_2)\sum_{k=0}^{\infty}(k+1)\mu' A(\boldsymbol{t})^k\mu - \sum_{k=0}^{\infty}\mu' A(\boldsymbol{t})^k\mu.$$

The fourth line of the display is correct as printed.

17. **Page 245, Reference [8].** “Thoery” should read “Theory.”